Local relative trace formulas

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- ex : GL_n, SL_n, $SO(V)$, $U(V)$...

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Known for GL_n (Harris-Taylor, Henniart), $SO(V)$, $Sp(V)$ (Arthur) and $U(V)$ (Mok, Kaletha-Minguez-Shin-White)

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Conjecture (Sakellaridis-Venkatesh)

Assume the SL_2-factor is trivial. Then

$$L^2(X) = \int_{[\varphi]}^{\oplus} \mathcal{H}_{\varphi'}\, d\varphi$$

- φ varies over (tempered) Langlands parameter into ${}^L\mathbb{G}_X$
- φ' is the Langlands parameter of G associated to φ and $\mathcal{H}_{\varphi'}$ is a multiplicity-free sum of representations in $\Pi_{\varphi'}$ (may be zero)

Trace Formula

- For $f \in C(G)$, $R(f) = \int_G f(g)R(g)dg \curvearrowright L^2(X)$ convolution operator

$$R(f)\varphi(x) = \int_X K_f(x,y)\varphi(y)dy, \;\; K_f(x,y) = \int_H f(x^{-1}hy)dh$$

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- In general $R(f)$ is not trace-class. Arthur's idea : consider "truncated" trace

$$J^T(f) = \int_X K_f(x,x)u(x,T)dx$$

where $u(.,T)$ is the characteristic function of a compact set $\Omega_T \subset X$ covering X as $T \to \infty$.

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- Try to evaluate $J^T(f)$ as $T \to \infty$ in two different ways : geometric (orbital integrals) and spectral (characters).

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Theorem (Waldspurger)

For certain $f \in C(G)$, we have

$$\int_{\mathcal{C}_X(H)} J(h,f)dh = \int_{\mathcal{R}_X(G)} J(\pi,f)d\pi$$

- *$\mathcal{C}_X(H)$: space of conjugacy classes in H, $\mathcal{R}_X(G)$: space of (virtual) representations of G*
- *$J(h,f)$: weighted orbital integral, $J(\pi,f)$: weighted character*
- *Remark : Here $1 \in \mathcal{C}_X(H)$ is an atom*
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- This also works for unitary groups : $H = U(W) \hookrightarrow G = U(W) \times U(V)$.

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- In these three cases, able to establish a weak trace formula
- Application : set $m_i(\pi) = dim\ Hom_{G_i}(\pi, L^2(X_i))$. If π is a square-integrable representation of $GL_{2n}(F)$ and $\pi' = JL(\pi)$, then

$$m_1(\pi) + m_2(\pi') = m_0(\pi)$$