Algebraic Symmetries I

Just as we can factor

$$z^3 - 1 = (z-1)(z^2+z+1),$$

we can factor

$$z^5 - 1 = (z-1)(z^4+z^3+z^2+z+1)$$

It follows that each of the four numbers

$$z_i = \cos(2k\pi/5) + i\sin(2k\pi/5), \quad k = 1,2,3,4$$

satisfies the equation

$$z^4+z^3+z^2+z+1 = 0$$

Thus our algebraic interpretation of the five vertices of the regular pentagon as the five fifth roots of unity has destroyed the five-fold symmetry. We have distinguished one vertex, placing it at the point 1=1+0i. So we now have to look for a different kind of symmetry, that among the four remaining vertices, or better the four remaining roots.

There is a one obvious symmetry, that which interchanges z_1 and z_4 as well as z_2 and z_3. This is an *algebraic* as well as an *geometric* symmetry because it is just a matter of replacing each of the numbers by its complex conjugate

$$a+bi \longrightarrow a-bi$$

and

$$(a+bi)\times(c+di) = ac-bd+(ad+bc)i \longrightarrow ac-bd-(ad+bc)i = (a-bi)\times(c-di)$$

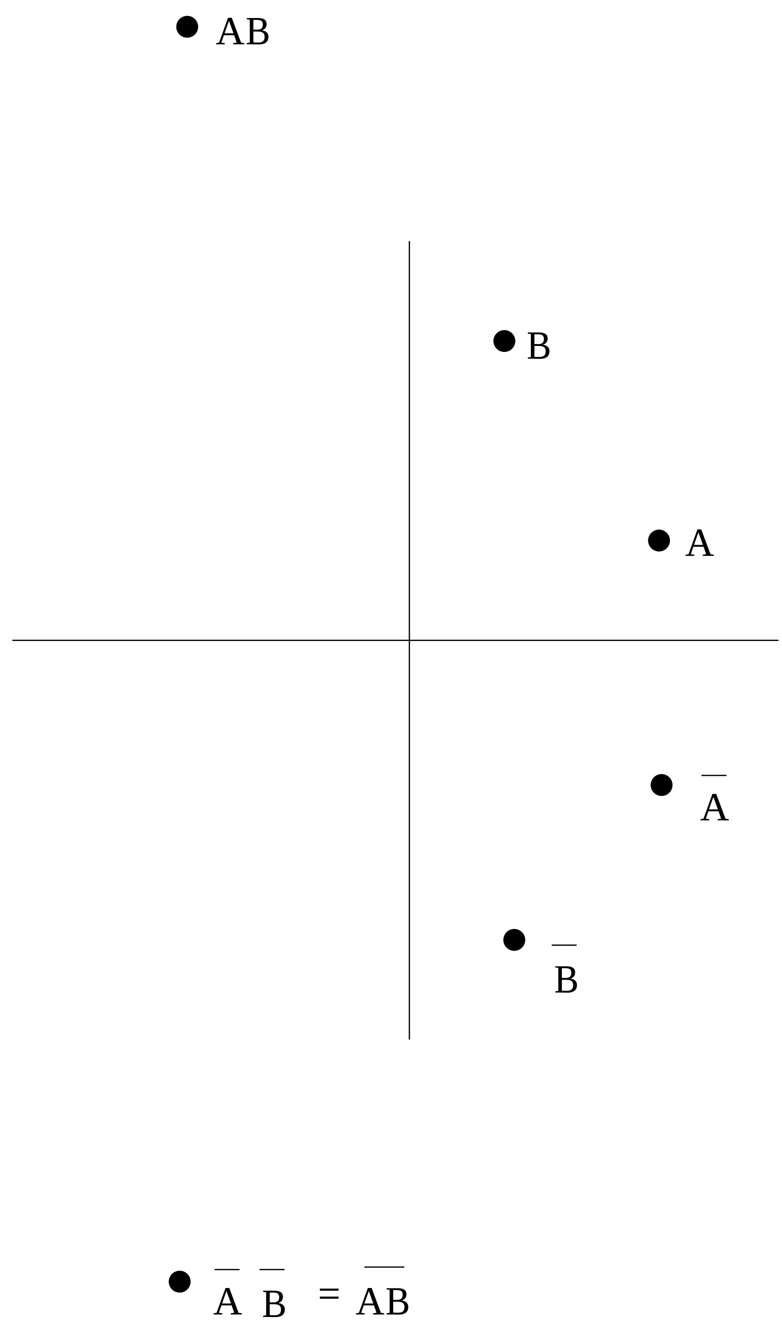
I take it as obvious that the complex conjugate of the sum or the difference of two complex numbers is the sum or the difference of their complex conjugates.

$$(a+bi)+(c+di) = (a+c)+(b+d)i \longrightarrow (a+bi)-(c+d)i = (a-bi)+(c-di)$$

In other words the operation of complex conjugation that interchanges z_1 and z_4 as well as z_2 and z_3

$$z_1 \longleftrightarrow z_4, \qquad z_2 \longleftrightarrow z_3$$

is like reflection in a mirror. All *arithmetic* properties are faithfully preserved.


AB
B
A
$\bar{A}$
$\bar{B}$
$\bar{A}\ \bar{B} = \overline{AB}$

Small remarks

It is sometimes useful to recall that the relation

$$z_1^4 + z_1^3 + z_1^2 + z_1 + 1 = 0$$

is the same as

$$z_4 + z_3 + z_2 + z_1 + z_0 = 0$$

Why is, for example, $z_1 z_2 = z_3$?

$$(\cos(\theta) + i \sin(\theta))(\cos(\varphi) + i \sin(\varphi) = \cos(\theta + \varphi) + i \sin(\theta + \varphi)$$

Take $\theta = 2\pi/5$ and $\varphi = 4\pi/5$. Then $\theta + \varphi = 6\pi/5$ and this becomes

$$z_1 z_2 = z_3$$

One shows in the same way that, for example, $z_2 z_2 = z_4$. Observe also that, along the same lines,

$$z_1^7 = (\cos(2\pi/5) + i \sin(2\pi/5))^7 = \cos(14\pi/5) + i \sin(14\pi/5)$$

and

$$\cos(14\pi/5) + i \sin(14\pi/5) = \cos(4\pi/5) + i \sin(4\pi/5) = z_2$$

This is because the angle $14\pi/5$ is equal to $2\pi + 4\pi/5$ and the cosine and sine do not change when 2π is added to or subtracted from an angle. Indeed, in some respects, the angle itself does not change! (Note: this statement is correct, but calls for some reflection!)

Algebraic Symmetries II

This is because i is just a symbol that stands for the square root of $-i$ and $-i$ is then introduced and defined by the condition that

$$i + (-i) = 0$$

But $-i$ is also just a symbol and can be taken as the primary symbol.* Then i is a second symbol that functions as $-(-i)$. Even if i is taken to have some meaning beyond that of a mere symbol, it cannot have a different meaning than $-i$, so that the two have to be regarded as perfectly interchangeable.

Are there other symmetries of this kind?

Whether there are other symmetries of this kind affecting all complex numbers is not a question for us, but we can ask whether there are symmetries of this kind affecting just z_1, z_2, z_3 and z_4. Before we do, we make use of the symmetry at hand. Since $z_1z_1 = z_2$ $z_1z_2 = z_3$, $z_1z_3 = z_4$, $z_1z_4 = 1$, $z_2z_2 = z_4$, $z_2z_3 = 1$ and so on, and since in addition

$$1 = -z_1 - z_2 - z_3 - z_4$$

the numbers

$$az_1 + bz_2 + cz_3 + dz_4,$$

where a, b, c and d are arbitrary ordinary fractions form a collection closed under addition, subtraction, multiplication, and even as it turns out division. The numbers that are equal to their own reflections can be singled out. These are the numbers

$$a(z_1 + z_4) + b(z_2 + z_3)$$

* The distinguishing characteristic of i is that $i^2 = -1$ but this characteristic is shared by $-i$.

The appearance of $\sqrt{5}$.

Let w be the number $z_1 + z_4$. It is equal to its own reflection. So is its square. Thus

$$w^2 = a(z_1+z_4)+b(z_2+z_3) = (a-b)(z_1+z_4)+b(z_1+z_2+z_3+z_4) = (a-b)w-b$$

Thus w satisfies a quadratic equation

$$w^2 + cw + d = 0, \qquad c = b - a, \quad d = b$$

We calculate this equation

$$w^2 = (z_1+z_4)^2 = z_1^2+z_1z_4+z_4z_1+z_4^2 = z_2+1+1+z_3 = 2-1-z_1-z_4 = 1-w$$

Thus

$$w^2 + w - 1 = 0 \qquad w = \frac{-1 \pm \sqrt{1+4}}{2}$$

Since w is a positive number,

$$w = \frac{-1+\sqrt{5}}{2}$$

In other words, w can as we know be constructed with ruler and compass. Since

$$z_1 + z_4 = w \Longleftrightarrow z_1 + \frac{1}{z_1} = w \Longleftrightarrow z_1^2 + 1 = z_1 w$$

we have

$$z_1 = \frac{w \pm \sqrt{w^2-4}}{2}$$

Since $w^2 = 1 - w$, this is

$$\frac{w \pm \sqrt{-3-w}}{2} = \frac{-1+\sqrt{5}}{4} \pm \frac{\sqrt{-\frac{5}{2} - \frac{\sqrt{5}}{2}}}{2}$$

Since z_1 lies above the axis of abscissas,

$$z_1 = \frac{-1+\sqrt{5}}{4} + i\frac{\sqrt{\frac{5+\sqrt{5}}{2}}}{2}$$

Symmetries III

Having found z_1, we can easily find z_4, its complex conjugate, and we can certainly find z_2 by squaring z_1. We can also find z_2 by working with $z_2 + z_3$ rather than w. This is, however, straightforward algebra. As Descartes insisted, the algebra often turns a problem into an almost unthinking manipulation of symbols, a turn that it can indeed often take, but we prefer another direction. This is the direction taken by Gauss.

Let ζ be the number z_1. Then $z_2 = \zeta^2$, $z_3 = \zeta^3$, and $z_4 = \zeta^4 = -1 - \zeta - \zeta^2 - \zeta^3$ because

$$(I) \qquad \zeta^4 + \zeta^3 + \zeta^2 + \zeta + 1 = 0$$

Our numbers $ez_1 + fz_2 + gz_3 + hz_4$ can also be expressed as

$$a + b\zeta + c\zeta^2 + d\zeta^3$$

$$a = -h, \quad b = e - h, \quad c = f - h, \quad d = g - h$$

Are these numbers all different?

This is an important question! The answer is yes and I shall give a proof following Gauss. We first note the consequences. Since the numbers are all different, we cannot have an equation

$$a + b\zeta + c\zeta^2 + d\zeta^3 = 0 = 0 + 0\zeta + 0\zeta^2 + 0\zeta^3$$

unless $a = b = c = d = 0$. This means that ζ satisfies the equation

$$(I) \qquad Z^4 + Z^3 + Z^2 + Z + 1 = 0$$

and essentially only this equation, because if we have any other such as

$$(II) \qquad Z^6 + AZ^5 + BZ^4 + CZ^3 + DZ^2 + EZ + F = 0$$

then by the process of long division we will have

$$Z^6+AZ^5+BZ^4+CZ^3+DZ^2+EZ+F = (Z^2+GZ+H)(Z^4+Z^3+Z^2+Z+1)+PZ^3+QZ^2+RZ+S$$

so that

$$P\zeta^3 + Q\zeta^2 + R\zeta + S = 0$$

We have just seen that this implies $P = Q = R = S = 0$. Thus

$$Z^6 + AZ^5 + BZ^4 + CZ^3 + DZ^2 + EZ + F = (Z^2 + GZ + H) \times (Z^4 + Z^3 + Z^2 + Z + 1)$$

and (II) is a consequence of (I).

Symmetries IV

This means that, from an abstract point of view, ζ is simply a number that satisfies the equation

$$\zeta^4 + \zeta^3 + \zeta^2 + \zeta + 1$$

and no other. But z_2, z_3 and z_4 have exactly the same property. Thus *for strictly algebraic purposes* we could take ζ to be z_2 rather than z_1. Thus z_1 is replaced by z_2. Since *all* algebraic relations are to be preserved, this entails replacing $z_2 = z_1^2$ by $z_2^2 = z_1^4 = z_4$ and $z_3 = z_1^3$ by $z_2^3 = z_1^6 = z_1$. Once again, because all algebraic relations are to be respected, the number

$$z_4 = z_1^4 = -1 - z_1 - z_1^2 - z_1^3$$

is to be replaced by

$$-1 - z_2 - z_2^2 - z_3^3 = z_2^4 = z_1^8 = z_1^3 = z_3$$

In general a number

$$a + b\zeta + c\zeta^2 + d\zeta^3 = a + bz_1 + cz_1^2 + dz_1^3 = (b-a)z_1 + (c-a)z_2 + (d-a)z_3 - az_4$$

is replaced by

$$a + bz_2 + cz_2^2 + dz_2^3 = a + bz_1^2 + cz_1^4 + dz_1 = (a-c) + (d-c)z_1 + (b-c)z_1^2 - cz_1^3$$

which is a number of the same kind. For example

$$z_3 = z_1^3 \longrightarrow z_1 \qquad z_4 = z_1^4 \longrightarrow z_2^4 = z_1^8 = z_1^3 = z_3$$

Forms of algebraic symmetries

The symmetry just examined can be viewed in two ways:

1) It takes the sequence $\{z_1, z_2, z_3, z_4\}$ of all roots of

$$Z^4 + Z^3 + Z^2 + Z + 1$$

to a sequence formed from the same numbers but in a different order

$$\{z_2, z_4, z_1, z_3\}$$

2) It takes any number

$$az_1 + bz_2 + cz_3 + dz_4 \qquad \text{(I)}$$

to a number

$$az_2 + bz_4 + cz_1 + dz_3 \qquad \text{(II)}$$

of the same kind.

This is the kind of symmetry that was later investigated in general by Galois. We have to spend some time growing accustomed to it. Suppose we apply the symmetry twice. Then

$$z_1 \to z_2 \to z_4$$
$$z_2 \to z_4 \to z_3$$
$$z_3 \to z_1 \to z_2$$
$$z_4 \to z_3 \to z_1$$

Thus applying the basic symmetry twice leads to the first symmetry considered, complex conjugation. We apply it again.

$$z_1 \to z_4 \to z_3$$
$$z_2 \to z_3 \to z_1$$
$$z_3 \to z_2 \to z_4$$
$$z_4 \to z_1 \to z_2$$

Yet again!

$$z_1 \to z_3 \to z_1$$
$$z_2 \to z_1 \to z_2$$
$$z_3 \to z_4 \to z_3$$
$$z_4 \to z_2 \to z_4$$

So the symmetry when repeated four times comes back where it began. It is a four-fold symmetry.

Anticipating Galois and his successors

No matter which of the numbers z_1, z_2, z_3, z_4 we take ζ to be, the collection of numbers

$$\text{(I)} \qquad a + b\zeta + c\zeta^3 + d\zeta^4, \quad a, b, c, d \text{ all fractions}$$

is the same. Modern mathematicians usually call the collection a field. the sum and the product of two numbers of this sort are again numbers of the same sort. This we have seen already. I give another example.

$$(1+\zeta)(1+\zeta^3) = 1 + \zeta + \zeta^3 + \zeta^4 = -\zeta^2$$

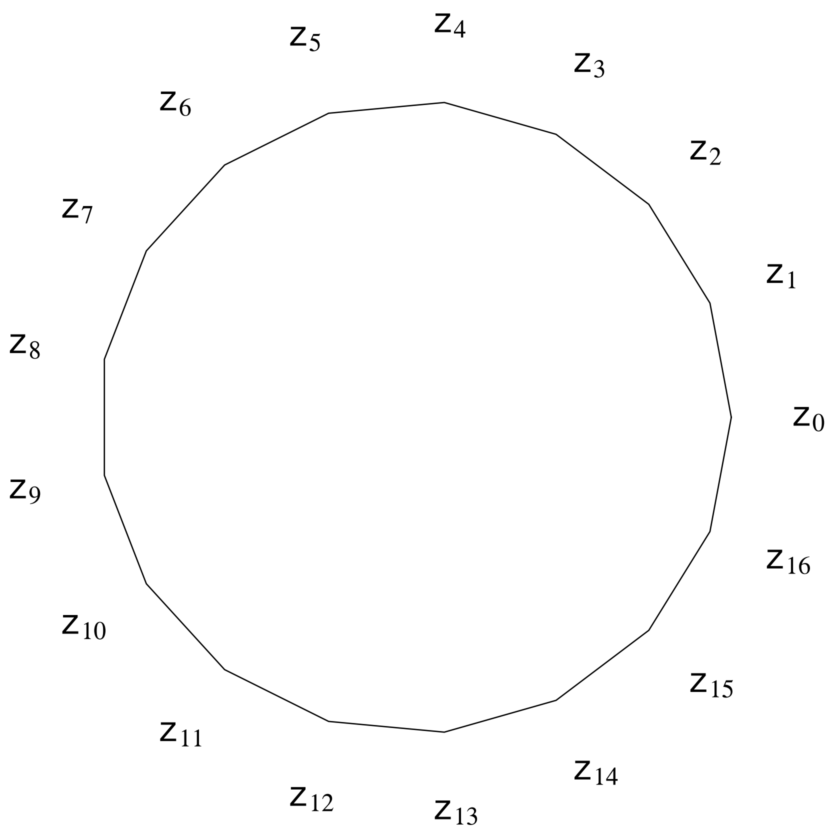
Any symmetry of this collection that respect the algebraic operations will take 0 to 0 and 1 to 1. Then adding and dividing it takes any fraction a/b to a/b. Moreover any root of

$$Z^4 + Z^3 + Z^2 + Z + 1 = 0$$

will be taken to another root. Thus z_1 will be taken to z_1, z_2, z_3 or z_4. In other words, the symmetry will be one of the four (including the trivial symmetry!) we already have. Denote the one taking z_1 to z_2 by the letter σ. Then the repeating σ to obtain $\sigma\sigma = \sigma$ we obtain the symmetry taking z_1 to z_4. Repeating again, we obtain $\sigma\sigma\sigma = \sigma^3$ which takes z_1 to z_3. Repeating again, we find that $\sigma\sigma\sigma\sigma = \sigma^4$ is the trivial symmetry.

Inside the collection of numbers there is a smaller collection of numbers that have a special symmetry. We met them before. They are those that are not affected by σ^2, thus by complex conjugation. They are the numbers $a+bw$, $w = z_1+z_4$. We were able to construct z_1 by succesive square roots, by first singling out this special collection of numbers, finding that any number in it satisfied a quadratic equation with fractions as coefficients, in particular that $w^2+w-1=0$, so that $w = \frac{(-1+\sqrt{5})}{2}$, and then solving $z_1^2 - z_1 w + 1 = 0$. We now apply these ideas, which I hope are clear, to the heptadecagon!

The Heptadecagon



$$z_k = \cos(2k\pi/17) + i\sin(2k\pi/17)$$

From the *Disquisitiones Arithmeticae*

There is a famous remark from the introduction to the seventh and last chapter of the *Disquisitiones* that I quote here. What it anticipates is the study of the division points on elliptic curves, in the remark a special elliptic curve, a study that led over the course of the nineteenth and twentieth century to many things, especially complex multiplication and l-adic representations, that are relevant to the Shimura-Taniyama-Weil conjecture,

Ceterum principia theoriae, quam exponere aggredimur, multo latius patent, quam hic extenduntur. Namque non solum ad functiones circulares, sed pari successu ad multas alias functiones transscendentes applicari possunt, e. g. ad eas, quae ab integrali

$$\int \frac{dx}{\sqrt{1-x^4}}$$

pendent, praetereaque etiam ad variam congruentarium genera: sed quoniam de illis functionibus transscendentibus amplum opus peculiare paramus, de congruentibus autem in continuatione disquisitionum arithmeticarum copiose tractabitur, hoc loco solas functiones circulares considerare visum est.

A proof by Gauss (beginning)

Recall that we want to show that z_1 is a root of the equation

I $$Z^4 + Z^3 + Z^2 + Z + 1 = 0$$

but of no equation of the forms

$$Z^3 + aZ^2 + bZ + c = 0 \tag{I}$$
$$Z^2 + aZ + b = 0 \tag{II}$$
$$Z + a = 0 \tag{III}$$

in which a, b c and d are fractions.

The impossibility of the last equation is clear because z_1 is not a fraction. If it were a root of the first, then using long division to divide (I) by $Z^3 + aZ^2 + bZ + c$, we would find

$$Z^4 + Z^3 + Z^2 + Z + 1 = (Z^3 + aZ^2 + bZ + c)(Z + d) + eZ^2 + fZ + g$$

Substitute z_1 to find that z_1 is also a root of

$$eZ^2 + fZ + g = 0$$

Then, as we just observed e is not 0, unless $e = f = g = 0$. If e is not 0, divide by it. Thus either

(IV) $$Z^4 + Z^3 + Z^2 + Z + 1 = (Z^3 + aZ^2 + bZ + c)(Z + d)$$

or z_1 satisfies an equation of type (II). If it satisfies (II), then perform a long division to obtain

$$Z^4 + Z^3 + Z^2 + Z + 1 = (Z^2 + aZ + b)(Z^2 + cZ + d) + eZ + d$$

Since $ez_1 + d$ cannot be 0 unless$e = d = 0$, we conclude that

(V) $$Z^4 + Z^3 + Z^2 + Z + 1 = (Z^2 + aZ + b)(Z^2 + cZ + d)$$

A proof by Gauss (continued)

We now show that the factorizations of (IV) and (V) are impossible. We first observe a very important fact.

An equation of degree n

$$Z^n + aZ^{n-1} + bZ^{n-2} + \ldots + d = 0 \tag{VI}$$

cannot have more than n roots!

Suppose (VI) has a root e. Using long division, divide by $Z - e$. The result is

$$Z^n + aZ^{n-1} + bZ^{n-2} + \ldots + d = (Z - e)(Z^{n-1} + AZ^{n-2} + \ldots + D) + f$$

Substitute e to see that $f = 0$. Now any other root e' not equal to e of (VI) must be a root of

$$Z^{n-1} + AZ^{n-2} + \ldots + D = 0, \tag{VII}$$

so that if (VI) had more than n roots, then (VII) would have more than $n-1$. All we have to do now is continue, working our way down to lower and lower degree until we arrive at an equation of degree one

$$Z + \alpha = 0$$

that clearly has only one root.

General comments

The proof we present is a proof that can be extended without too much additional effort to the following statement.

Suppose p is a prime. Then the polynomial

$$Z^{p-1} + Z^{p-2} + Z^{p-3} + \dots Z + 1$$

admits no factorization.

This is a statement proved by Gauss in the *Disquisitiones.* We will need it for $p = 17$.

According to Bourbaki's *Éléments de l'histoire des mathématiques*, this was the first general statement of this sort about polynomials ever proved. It seems to me that in one sense, it is also the last. In the nature of things, there are very few large classes of polynomials that are all irreducible. On the other hand, if taken as a statement in the sense of Galois theory, as a statement affirming that the Galois group of the equation is large, then there are indeed other statements of this sort available and perhaps many more yet to be proved. The size and nature of Galois groups is a reasonably important mathematical topic, at least for mathematicians.

Since the proof in the general case is not so different from the proof for $p = 5$, we can expect a certain amount of sophistication.

A proof by Gauss (continued)

Suppose we had a factorization (IV). Since z_1, z_2, z_3 and z_4 are all roots of

$$Z^4 + Z^3 + Z^2 + Z + 1 = 0$$

each of them is either a root of

$$\text{(VIII)} \qquad Z^3 + aZ^2 + bZ + c = 0$$

or of

$$Z + d = 0$$

Since the latter is impossible, because $-d$ is, in contrast to z_1, z_2, z_3 and z_4, a fraction, they are all roots of (VIII) which gives (VIII) one root too many.

Thus we must have

$$\text{(V)} \qquad Z^4 + Z^3 + Z^2 + Z + 1 = (Z^2 + aZ + b)(Z^2 + cZ + d)$$

The first thing is to establish that a, b, c and d are all integral. For this we need a little number theory. Suppose that one of these numbers is not integral. Then it is of the form

$$\frac{m}{n}, \qquad n \text{ positive}, \qquad n > 1$$

in which m and n have no common divisor. To be explicit, we write

$$\frac{3}{30}$$

as

$$\frac{1}{10} = \frac{1}{2 \times 5}$$

Thus there is some prime number p such that $n = rp^t$, $t > 0$, p does not divide r. For example, if $m/n = 1/10$ then p could be 5 and we would have $m = 1$, $r = 2$, $t = 1$.

We write each of a, b, c and d in this way.

$$a = \frac{a_1}{a_2 p^q}, \quad b = \frac{b_1}{b_2 p^r}, \quad c = \frac{c_1}{c_2 p^s}, \quad d = \frac{d_1}{d_2 p^t},$$

Suppose that at least one of the numbers q, r, s, t is positive. If, for example, $q = 0$ then p might divide a_1. Thus suppose that not all of a, b, c and d are integral. We multiply (IV) out to obtain

$$Z^4 + (a + c)Z^3 + (b + ac + d)Z^2 + (bc + ad)Z + bd = 0.$$

First of all, $a + c = 1$ is an integer. This is possible only if $q = s$, for if q is not equal to s then one of them is positive and larger than the other. The denominator of $a + c$ would contain the factor p^n if n is the larger of q and s. For example

$$\frac{2}{3 \times 5^2} + \frac{3}{5} = \frac{2 + 9 \times 5}{3 \times 5^2} = \frac{47}{3 \times 5^2}$$

We conclude that $q = s$.

Looking at $b + ac + d$, we next conclude that either $r = t$ or $q + s$ is equal to one of r or t. This line of argument quickly becomes confusing. The best thing is to follow Gauss and to prove a general theorem. Oddly enough, it is an argument that can best be explained in the general case. Here is what we want to show.

Suppose that the product of

$$Z^n + a_1 Z^{n-1} + a_2 Z^{n-2} + \ldots + a_{n-1} Z + a_n$$

and of

$$Z^m + b_1 Z^{m-1} + a_2 Z^{b-2} + \ldots + b_{n-1} Z + b_m$$

has integral coefficients and that all of the numbers a_i and b_j are fractions (and not some more complicated kind of irrational number!). Then all of the numbers a_i and b_j are integers.

We suppose not and choose a p that divides the denominator of at least one a_i or one b_j. Then we write

$$a_i = \frac{m_i}{n_i p^{r_i}} \qquad b_j = \frac{s_j}{t_j p^{s_j}}$$

It is understood that p does not divide n_i and that it does not divide t_j. Starting with r_1, examine all the r_i and let r_k be the first that is at least as large as all the others. Thus r_k is bigger than those that came before and at least as large as those that come after. In the same way s_l is to be larger than the s_j that come before and at least as large as those that come after.

The product is of degree $m+n$. we look at the coefficient of the power $Z^{m+n-k-l}$. It is

$$a_{m+n-k-l}b_0 + a_{m+n-k-l-1}b_1 + a_{m+n-k-l-2}b_2 + \ldots + a_0 b_{m+n-k-l}$$

Some of these terms are purely fictive, those in which a_i appears with i larger than n or b_j with j larger than m. If a_0 or b_0 appear they are taken to be 1. For example, if $n=5$, $m=3$ then the coefficient of Z^4 is

$$a_4 + a_3 b_1 + a_2 b_2 + a_1 b_3$$

In general, the coefficient of Z^{k+l} is supposed integral and contains the term $a_k b_l$ in whose denominator $p^{r_k+s_l}$ occurs. It also contains terms like $a_{k-1}b_{l+1}$ whose denominator contains at most the factor $p^{r_l-1+s_l}$ or like $a_{k+1}b_{l-1}$ whose denominator contains at most the factor $p^{r_k+s_l-1}$. Thus when we put everything over a common denominator, we will have a $p^{r_k+s_l}$ in the denominator that cannot be removed. This is a contradiction.

An example.

$$(Z^3 + \frac{1}{3}Z^2 + \frac{2}{9}Z + \frac{1}{9}) \times (Z^2 + \frac{4}{3}Z + \frac{7}{3})$$

Here $k=2$, $l=1$, so that $m+n-k-l=2$. The coefficient of Z^2 in the product is

$$\frac{1}{3}\frac{7}{3} + \frac{2}{9}\frac{4}{3} + \frac{1}{9} = \frac{1}{3} + \frac{8}{27} + \frac{1}{9} = \frac{21}{27} + \frac{8}{27} + \frac{3}{27} = \frac{32}{27}$$

Final step

We now know that in (V) the four numbers a, b, c and d are integral. The argument we used was taken from §42 of the *Disquisitones*, thus from an early chapter. The next part is from §341, thus from a very late, indeed the last chapter, the one devoted to the division of the circle, thus to regular polygons. The number z_1 will be a root of

$$Z^2 + aZ + b = 0$$

or of

$$Z^2 + cZ = d = 0$$

We can suppose that it is a root of the first. Then z_4, its complex conjugate will also be a root, so that the roots of the second are z_2 and z_3.

The polynomial $Z^2 + aZ + b$ is divisible by $Z - z_1$

$$Z^2+aZ+b = (Z-z_1)(Z-z_4) = Z^2-(z_1+z_4)Z+1 = Z^2-2\cos(2\pi/5)Z+1$$

which we write as

$$Z^2-2\cos(2\pi/5)Z+\cos^2(2\pi/5)+\sin^2(2\pi/5) = (Z-\cos(2\pi/5))^2+\sin^2(2\pi/5)$$

In the same way,

$$Z^2 + cZ + d = (Z - \cos(4\pi/5))^2 + \sin^2(4\pi/5)$$

Thus if Z is an ordinary real number, both $Z^2 + aZ + b$ and $Z^2 + cZ + d$ are positive. If Z is an integer, they are integers.

Since

$$(Z^2 + aZ + b)(Z^2 + cZ + d) = Z^4 + z^3 + Z^2 + Z + 1$$

we have first of all, for $Z = 1$,

$$(1 + a + b)(1 + c + d) = 5$$

Thus one of these numbers is 1 and the other is 5. On the other hand,

$$Z^2 + aZ + b + Z^2 + cZ + d = Z^4 - (z_1 + z_2 + z_3 + z_4)Z + 2 = 2Z^4 + 3$$

which is 5 when $Z = 1$. We conclude that $1 + 5 = 5$, which is out of the question.

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LITERARISCHE NACHRICHTEN.

I. Französische Literatur.

Zweyte Uebersicht.

Die für Paris bestimmten Mitglieder des grossen *Institut national* sind zwar in der letzten authentischen Nachricht aus Paris (S. *Intelligenzblatt* n. 19. S. 153 f.) schon so genau als möglich angegeben worden. Es dürfte aber in mehr als einer Rücksicht nicht uninteressant seyn, auch bey jeder Classe und Section die Namen derer bekannt zu machen, die bloss *praesentirt* worden sind. Auch unter diesen sind viele im Auslande berühmte und geschätzte Namen, deren Verzeichniss hier wenigstens als Antinekrolog dienen, und zu mancherley Betrachtungen Anlass geben kann. So findet man z. B. unter diesen auch die Namen der Hr. v. *Villoison*, des Grafen *Gorani* und des einst mit Schrecken genannten *Hassenfratz*. Auch sind hier und da noch einige Berichtigungen hinzuzufügen:

I Klasse.

1. *Mathematik.* *Präsentirt* Lacroix.
2. *Mechanik.* Caroché, Molard, Bréguet, *Berthout*, der zum wirklichen Mitgliede gewählt wurde, unterscheidet sich durch den Vornamen *Ferdinand*.
3. *Astronomie.* Le François.
4. *Experim. Phys.* Deparcieux, Carnot.
5. *Chimie.* Hassenfraz, Seguin, Baumé, Cadet.
6. *Natur-Hist. und Mineral.* Sage, Welter, Gillet l'Aumont, Faujas.
7. *Botanik.* La Billardiére, Palissot de Beaurois.
8. *Anatomie* und *Zoologie.* a) Pinel, Sue. b) Brugnieres, Geoffroi, Olivier, Rose.
9. *Med. u. Chirurg.* a) Thouret, Cayes, la Tisse, Andry, Corvisar. b) Deschamps, Chaussier, Suë.
10. *Oeconomie u. Thierarzney.* a) André Michaut, Creté Palluel, Dubois.

II Klasse.

1. *Analyse der Empf.* Segond, La Romignière, La Salle.
2. *Moral.* La Bène, Villeterque, Blavet, Dinger, Rétif de la Bretonne, Ricard, Gorani.
4. *Gesetzgebung.* Treilhard.
5. *Polit. Oecon.* Jolivet, Otto: Parcos.
6. *Geschichte.* Garnier, Du Theil.
7. *Statistik u. Geographie.* Barbier du Bocage, Bélème, Bory, Rayneval, Bourgoin, Otto.

III Klasse.

1. *Grammatik.* Guerou, Lohier, Pougens, Binet, Marmontel, Palissot.
2. *Alte Sprachen.* Gail, Larcher, Voilloison, Maltor.
3. *Poësie.* Andrieux, Sedaine, Cailhava.
4. *Alterthümer.* Barthelemy, Millin.
5. *Mahlerey.* Gerard, Suvée, Giraudet.
6. *Bildhauerey.* Boizot, Gois.
7. *Baukunst.* Brognard, Molinos.
8. *Tonkunst.* Langlé, Le Sueur, Cherubini, Martini.
9. *Deklam.* Talma, La Rive.

Noch ist hierbey anzumerken, dass der berühmte Dichter *Delille*, aus dessen mit Sehnsucht erwarteten Gedicht *uber die Einbildungskraft* so vortrefliche *Morceaux* im *Journal Encyclopedique* neuerlich gelesen worden sind, in einem mit ziemlicher Bitterkeit geschriebenen Brief sich für unfähig erklärt hat, die ihm zugedachte Mitgliedschaft anzunehmen. *Er* kann sich, wie es scheint, bey seiner erklärten Vorliebe für die alte Regierung noch nicht mit den neuen Formen aussöhnen. Der durch seine trefflichen Schauspiele, besonders durch den *vieux Célibataire* belobte und beliebte *Collin d'Harleville* war in der Section der Grammatik statt *Gurots*, der die Ernennung verbeten hatte, vom Directorium bestimmt worden. Als er von dieser Stelle Besitz nehmen wollte, kam die Section der Dichter, und bewies, dass er ihnen angehöre, und dass sie ihn schon zu ihrem Mitgliede erwählt hatten. Zwiefach geehrt durch diese doppelte Ernennung folgte er dem Rufe seiner nahern Mitbrüder, der Dichter, und trat in ihre Section ein. Und daran that er ganz recht, sagt ein Sprecher der öffentlichen Meinung, *un bon poëte fait plus qu' cindier et enseigner la langue: il la crée et l' enrichit.* Diess wollten einst die Väter des Wörterbuchs der Academie nicht Wort haben.

Den 1sten Nivose (21. Decembr. 1795) hielten die 144 bis jetzt ernannten und bestätigten Mitglieder ihre *Seance d'ouverture*, die *Dusaulx* als Präsident nach dem Alter, durch Vorlesung des Gesetzes, wodurch das Institut seine Constitution erhält, eröffnete. *Delisle de la Salle* panegyrisirte hierauf die neue Einrichtung mit einer ziemlich ermüdenden Weitschweifigkeit. *Fourcroy* und *La Lande* thun Vorschläge wegen der Ernennung einzelner Ausschüsse, die von *Chenier* genauer bestimmt werden. *Laplace* dringt darauf, dass die Arbeiten der einzelnen unter-

(3) U

Menschen, oder Moloch. den Minister mehr bewundern solle. Die *Deckersche* Buchhandlung in Basel hat, wie verlautet, die Uebersetzung dieser Schrift einem sachkundigen Manne aufgetragen, der sie noch durch einige Zusätze vervollständigen wird.

In Rücksicht auf antiquarische und historische Forschungen ist *Dupuis Origine de tous les cultes, ou religion universelle*, wovon 2 Ausgaben, eine in drey Quartbänden, die andere in 12 Octavbänden, zu einer jeden ein Bändchen Kupfer, erschienen sind, unbezweifelt das wichtigste Werk, das nach Barthelemys *Anacharsis* in Frankreich erschienen ist. *Dupuis*, ein Schüler und Liebling des grossen *La Lande*, hat die ganze Mythologie der alten Welt auf die neuesten Resultate der Sternkunde zu gründen, und dabey eine weit festere Basis, als *Court de Guebelin* und andere vor ihm, zu finden gewusst. Indem er den Zodiacus umdreht, und so zum richtigsten Ackerkalender der Aegypter macht, indem er ferner die bekannte Erfahrung des Fortrückens um ein ganzes Himmelszeichen in 2151 Jahren sehr geschickt auf die Verwirrung dieser Himmelshieroglyphen anwendet, wird allerdings in der ägyptischen Sterntheologie alles hell und deutlich. Wer unsers *Gatterers* Ideen, besonders einige seiner Vorlesungen in den *Comment. Soc. Goetting.* fleissig studiert hat, wird auch bey *Dupuis* bald zu Hause seyn, ohne jedoch die grundlosen Folgerungen für die spätere griechische Mythologie, die hier viel zu sehr astronomisirt wird, zu unterschreiben.

Die neueste Reisebeschreibung von einiger Bedeutung ist *Voyage de deux Français en Allemagne, Danemark, Suede, Russie et Pologne fait en 1790-92.* 6 Vol. in 8. (10 liv. numer.) Bey aller Oberflächlichkeit werden die Anekdotenliebhaber hier doch, besonders in den letzten 3 Theilen, wo von Schweden, Russland und Polen die Rede ist, ihre Rechnung sehr gut finden. Merkwürdig ist der Umstand: Auf der ganzen Reise besuchten die Herrn Voyageurs nur einen Mann, der nicht zum Hofe gehörte, *Klopstock*, und von diesem sprechen sie nicht einmal mit Achtung. Das Buch muss also auch in Deutschland in allen den Zirkeln grosses Glück machen, wo die Gelehrsamkeit verboten und der Gelehrte im Bann ist.

Die neuesten Romane, die auch wohl ausser Frankreich den Liebhabern dieser Leserey einige Befriedigung gewähren können, sind *Lettres de deux amans, habitans de Lyon*, par le cit. *Leonard.* 2 Vol. in 18. gehört zur empfindsamen Klasse, und kann mitunter erschüttern; *Illyssle, ou la beauté outragée par elle même*, ein Feenmährchen, das *starken* Glauben fodert; und *les trois soeurs*, par Madame *Beurnon-Mallarme.* 4 Vol. in 18. ein ächtfranzösisches Machwerk, mit anglisirten Namen.

Das neueste gute Lustspiel wurde auf dem Theater der Republik gegeben, und ist von *Picard: Les amis de College, ou l'homme oisif et l'Artisan.* 3 Acte in Versen. Drey Schulfreunde haben sich gegenseitige Unterstützung angelobt, und der eine, ein Dichter, kommt nun in den Fall, sie wirklich von den beyden andern zu fodern. Ein gut angelegter Plan, reich an glücklich benutzten Situationen.

Bey einem Ueberblick der neuesten franz. Literatur darf selbst das *äussere* Frankreich nicht ganz übersehn werden. Unter den in der A. L. Z. noch nicht angezeigten Werken dieser Art sind die *Memoires sur la vie et le caractere de Mad. la Duchesse de Polignac avec des Anecdotes interessantes sur la revolution Française et la personne de Mar. Antoinette.* London, Debrett. 1796. (2 sh. 6 d.) besonders merkwürdig. Sie werden auf dem Titel der Diane von Polignac selbst zugeschrieben.

Zu den ältern in Paris erscheinenden Zeitschriften gesellen sich jetzt zwey neue: *le Courrier des Enfans*, eine Kinderschrift, in der *Berquin's* Leichtigkeit glücklich nachgeahmt ist. Von ihr sind bis jetzt 4 Hefte herausgekommen. Und ein grammatisches, durch den revolutionairen Sprachunfug doppelt nöthig gewordenes Werk: *Journal de la langue Françoise* von *Domairgue*, der Mitglied des Nationalinstituts ist, und *Thurot*, dem Uebersetzer von *Harris's philosophical Grammar.* Er zerfällt in 3 Abtheilungen. Die erste löst grammatische Fragen auf. Die zweyte giebt einen *Cours suivi de la langue*, stellt Musterstücke aus den franz. Classikern auf, und beurtheilt sie.

II. Beförderungen.

Der bisherige Professor der Rechte zu Altorf, Hr. Dr. *Emminghaus*, ist zum ordentlichen Professor der Rechte nach Erlangen berufen worden, und hat diesen Ruf angenommen.

Ebendaselbst ist dem Hn. Dr. C. *Gross*, aus Urach, ehemaligem Instructor der Prinzen von Würtemberg, Vf. der mit Beyfall aufgenommnen *Geschichte der Verjährung nach römischem Rechte*, eine ordentliche Professur der Rechte ertheilt worden.

III. Neue Entdeckungen.

Es ist jedem Anfänger der Geometrie bekannnt, dass verschiedene ordentliche Vielecke, namentlich das Dreyeck, Viereck, Funfzehneck, und die, welche durch wiederholte Verdoppelung der Seitenzahl eines derselben entstehen, sich geometrisch construiren lassen. So weit war man schon zu Euklids Zeit, und es scheint, man habe sich seitdem allgemein überredet, dass das Gebiet der Elementargeometrie sich nicht weiter erstrecke: wenigstens kenne ich keinen geglückten Versuch, ihre Grenzen auf dieser Seite zu erweitern.

Desto mehr, dünkt mich, verdient die Entdeckung Aufmerksamkeit, dass *ausser jenen ordentlichen Vielecken noch eine Menge anderer, z. B. das Siebenzehneck, einer geometrischen Construction fähig ist.* Diese Entdeckung ist eigentlich nur ein Corollarium einer noch nicht ganz vollendeten Theorie von grösserm Umfange, und sie soll, sobald diese ihre Vollendung erhalten hat, dem Publicum vorgelegt werden.

C. F. Gauss, a. Braunschweig,
Stud. der Mathematik zu Göttingen.

Es verdient angemerkt zu werden, dass Hr. Gauss jetzt in seinem 18ten Jahre steht, und sich hier in Braunschweig mit eben so glücklichem Erfolg der Philosophie und der classischen Litteratur als der höhern Mathematik gewidmet hat.

Den 18 April 96.

E. A. W. Zimmermann, Prof.

IV.

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Den 18 April 96.

E. A. W. Zimmermann, Prof.

Die Allgemeine Literatur-Zeitung

From Meyers Enzyklopädisches Lexikon:

This review appeared in Jena from 1785 to 1803 and in Halle from 1804 to 1849 and was a leading organ of German classical and romantic literature. Goethe, Schiller and Kant were among the editors and authors. It moved to Halle as a result of an effort of the romantics to increase their control over the review and was replaced in Jena, on the initiative of Goethe by the Jenaische Allgemeine Literatur-Zeitung.

Gauss

Gauss, who was born in 1777, wrote the *Disquisitones* between 1796 and 1798, thus between his nineteenth and twenty-first years. It did not appear until 1801. It contains a great deal in the way both of theorems and theories, the most important being:

1. Proof of law of quadratic reciprocity. The statement was already known at the time, but even the best of the eighteenth century mathematicians were unable to find a proof. It remains a central mathematical theorem.

2. Developed the theory of binary quadratic forms. In particular, he introduced the notion of composition of quadratic forms and established its properties. Although in some respects, namely in the context of the notion of ideal number, composition has become a common working tool of all algebraists and number-theorists, Gauss's theory itself is still difficult and little known. His form of the theory would appear to be that best suited to computation.

3. Cyclotomic fields. The construction of the regular heptadecagon, and, more generally, the analysis of the numbers formed from roots of unity. This is thus one of the earliest manifestations of Galois theory. Gauss presumably knew more than he included in the book, but he, apparently, published very little more on the subject.

His thesis of 1799 established, in effect and for the first time, that every polynomial equation has a root.

$$Z^n + aZ^{n-1} + bZ^{n-2} + \ldots + d = 0$$

As the root may be complex and the thesis did not refer to complex numbers, the formulation of the thesis was necessarily somewhat different. I have already emphasized that this is a basic mathematical fact.

These are all theories and results to which the contributions of Gauss are clear. In addition, he appears to have occupied himself as a very young man, even as an adolescent, with other important problems and observations, for example, with the nature of geometries in which the parallel axiom of Euclid is not satisfied and with the arithmetic-geometric mean, which is both an elementary and an advanced topic. Here, however, the evidence is different. It consists of Gauss's recollections as a somewhat saturnine older man, so that it appears to be difficult to disentangle what he himself discovered later, what he learned from other authors as an adolescent – he had an early and extensive acquaintance with the work of various leading eighteenth century mathematicians – as well as what he learned later, from what he discovered early. One could certainly spend a lot of time with Gauss's Collected Works and with his correspondence, reflecting on these matters.

One extremely useful reference is a diary that Gauss kept between 1796 and 1814, with 146 entries that record his principal discoveries. The lemniscate function to which, as I noticed, Gauss alludes in the first lines of chapter 7 of the *Disquisitiones* appears several times. I was asked what Gauss what may have meant with his allusion to the lemniscate. A brief examination of the diary and of the collected works, which contain ample editorial comments, makes perfectly clear what Gauss knew. Although it is a digression from my main purpose, it is worthwhile to tarry a little on the matter.

The goal of this year's lectures is after all to communicate, starting at the beginning, some genuine mathematical understanding, beyond the gee-whiz or what I refer to as the Jack Horner manner, of recent achievements in number theory, and it would be a shame, since we are now in a position to appreciate a more detailed explanation of Gauss's allusion, to let slip the occasion of acquiring more concrete information.

Gauss was an overwhelming presence in nineteenth century mathematics, Even though he exerted little active influence. The German pre-eminence in mathematics as a whole, and in certain domains such as number theory in particular, that lasted throughout the nineteenth century and until the early thirties is due in good part to him, although the Prussian university system was probably also a significant factor. I have not studied these matters. Oddly enough it appears to have been André Weil who was most thoroughly imbued with the aspirations of German number theory, both through direct, personal experience as a young man during the twenties and through his studies of variou nineteenth century authors. Simplifying, for the purposes of brevity, an elaborate development in which a large number of mathematicians took part, one might say that he not only brought it intact through the war at its highest level but also was the principal source of its transformation into the theories that were finally and successfully exploited in the proof of Fermat's theorem.

His major contribution was, oddly enough, a set of conjectures, the Weil conjectures, now demonstrated, but by others. He is, in various comments to his papers, quite explicit about the relation of these conjectures to Gauss and the lemniscate.

I quote from Weil's *Two lectures on number theory, past and present*, an essay I recommend to your attention.

"In 1947, in Chicago, I felt bored and depressed and, not knowing what to do, I started reading Gauss's two memoirs on biquadratic residues, which I had never read before. . . . This led me in turn to some conjectures."

A few lines later Weil draws attention to the very last entry in Gauss's diary, an entry to which we shall come in a moment.

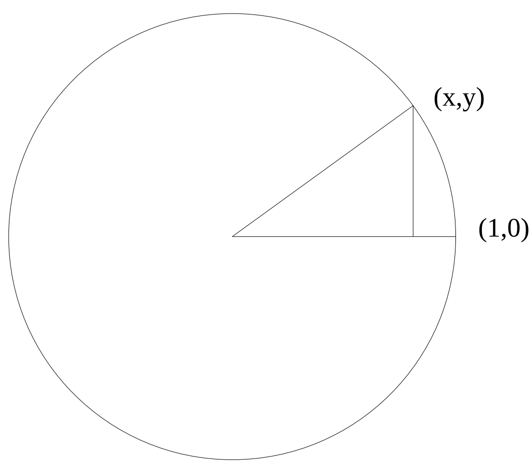
$$Z^n + aZ^{n-1} + bZ^{n-2} + \ldots + d = 0$$

Digression

The digression at first sight seems to demand some knowledge of the calculus, but it does not. I first write down a formula that may be familiar to some, but not to all. No matter! Do not puzzle over the left side. It is no more than the mathematician's usual fastidious way of writing down the length of the arc from the point $(1,0)$ on the circle to the point (x,y) and that is what we mean by θ, which of course has to be measured in radians, thus in units in which the radius is 1, but that is the unit chosen. If

$$(A) \qquad \int_0^y \frac{dt}{\sqrt{1-t^2}} = \theta,$$

then $y = \sin(\theta)$. If $y = 1$ then $\theta = \pi/2$.



We now do something similar for the lemniscate, a curve defined by the equation

$$(x^2 + y^2)^2 - (x^2 - y^2) = 0$$

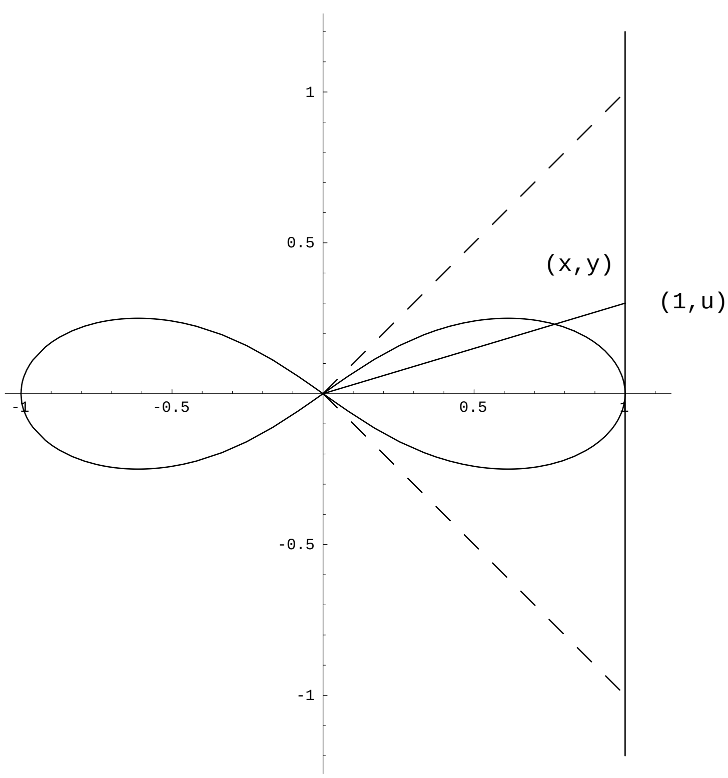
The lemniscate is the curve in the form of a bow. The point (x, y) is the point where the line through the points $(0, 0)$ and $(1, u)$ cut the curve. The length of the curve from $(1, 0)$ to the point (x, y) is expressed mathematically as

$$(B) \qquad \int_0^u \frac{dt}{\sqrt{1-t^4}} = \theta.$$

Once again, there is no need to be troubled by the integral. It is again just a way of expressing the length of an arc of a curve. Observe that θ plays here the role of the angle in a circle measured in radians. Gauss wrote

$$u = \text{sin lemn}(\theta).$$

If $u = 1$, then θ is some number that I call, following Gauss, $\varpi/2$. Thus $\varpi/2$ is the length of the upper loop on the right running from $(1, 0)$ to $(0, 0)$.



$$x = \frac{\sqrt{1-u^2}}{1+u^2}$$

$$y = \frac{u\sqrt{1-u^2}}{1+u^2}$$

Constructing a regular triangle, a regular pentagon, or a regular heptadecagon is the problem of dividing the total circumference of the circle into three, five or seventeen arcs of equal length. We could consider the same problem for the lemniscate, taking the initial point, which is now important as it was not, because of symmetry, for the circle, to be the point $(0, 1)$. In an entry for March 19, 1797 Gauss notes that this leads to an equation for u of degree m^2, whereas for the circle it was an equation of degree m. In the cases already considered, m was 3 or 5. We remove one easy root, $u = 0$ corresponding to the first point of division. This leads to equations of degree $m - 1$ or for a lemniscate $m^2 - 1$. In a later entry, apparently for April 15, he observes there is a problem of separating the real roots of this equation from the complex. He is seeking the real roots. The corresponding equation for the circle, thus for y, has only real roots. It is the numbers $x + iy$ that are complex. The real roots of the equation of degree $m^2 - 1$ give the division points and there are $m - 1$ of them.

In an entry dated March 21, he observes implicitly (all the entries are cryptic) that these $m - 1$ roots are numbers that can be constructed with a ruler and compass.

Lemniscata geometrice in quinque partes dividitur.

The young Norwegian mathematician Abel published in 1827 and 1828 proofs of the assertions implicit in Gauss's remark in the *Disquisitiones*. For $m = 3, 5$ the necessary constructions had been found much earlier, by 1750, by the Italian geometer Count Fagnano

The connections of the lemniscate with number theory are too ramified for us to discuss them in any more detail. The division points were important then and are important now, but so are what are called congruences modulo a prime. The two topics are very closely related. I end with the entry to which Weil alluded, the very last in the diary, dated July 9, 1814.

[III.]

[TEILUNG DER LEMNISKATE.]

[Eintragungen im LEISTE.]

[1.]

[S. 69]

Die Theilung der Lemniscata in sieben Theile gibt die Gleichung:

$$16\,(1-x^4)\left(\frac{1\quad -5\quad -5\quad +1}{1\quad +20\quad -26\quad +20\quad +1}\right)^2 = \left(\frac{3\quad * -6\quad * -1}{1\quad * +6\quad * -3}\right)^2.$$

[2.]

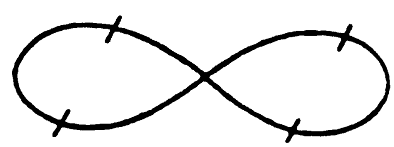
[S. 87]

Sit $\sin \frac{1}{5} k R = (k)$, tum habebuntur aequationis radices (0), $\pm(4)$, $\pm(2)$,

$$\frac{(0)+(5)(4)(1)\sqrt{-1}}{(1)-(0)(5)\ldots}$$

$$\frac{(4)+(1)(4)(1)\sqrt{-1}}{(1)-(4)(4)(1)\sqrt{-1}}$$

$$\frac{(2)-(3)(4)(1)\sqrt{-1}}{(1)+(2)(3)(4)\sqrt{-1}}$$



$$\pm 1,\quad \sqrt{-1}\left[\begin{matrix}(0)\\ \hline (2)\\ \hline (4)\end{matrix}\right|\quad \left.\begin{matrix}\dfrac{(4)}{(1)}\cdot\dfrac{1+(1)(1)\sqrt{-1}}{1-(4)(4)\sqrt{-1}}\\[2ex] \dfrac{(2)}{(3)}\quad\dfrac{1+(3)(3)\sqrt{-1}}{1-(2)(2)\sqrt{-1}}\end{matrix}\right|$$

[3.]

[S. 90—91]

1	1,31102877 $\left[=\frac{\varpi}{2}\right]$
2	1,71879545
4	2,95416
5	3,87311
6	5,07777
8	8,72765
9	11,44320
10	15,00

[cos 18°]

0,965425785 [$= \mu\, 18^0$]
1,034180311 [$= \nu\, 18^0$]

sin 36°

$$
\begin{array}{rl}
0{,}524411511 & \left[=\frac{\varpi}{5}\right] \\
-\quad 661011 & \\
-\quad 292 & \\
\hline
0{,}523750208 & [= M\, 36^0] \\
1{,}006302208 & \left[= 1 + \frac{1}{12}\left(\frac{\varpi}{5}\right)^4\right] \\
-\quad 567 & \\
\hline
[1{,}006301641 & = N\, 36^0]
\end{array}
$$

[sin] 36° =

$$\frac{0{,}523750208}{1{,}006301641} = 0{,}5204703904.$$

[sin] 72° = cos 18°

$$\frac{0{,}965425785}{1{,}034180311} = 0{,}9335177577.$$

[4.]

[S. 102]

Die Theilung der Lemniscata in 5 Theile führt auf diese Gleichung

$$\frac{9-36x^4+30x^8+12x^{12}+x^{16}}{1+12x^4+30x^8-36x^{12}+9x^{16}} = \frac{4(1-x^4)}{1+2x^4+x^8}$$

$$
\left.\begin{array}{rrrrrrr}
9 & -36 & +30 & +12 & +1 & & \\
 & +18 & -72 & +60 & +24 & +2 & \\
 & & +9 & -36 & +30 & +12 & +1 \\
-4 & -48 & -120 & +144 & -36 & & \\
 & +4 & +48 & +120 & -144 & +36 & \\
\hline
5 & -62 & -105 & +300 & -125 & +50 & +1
\end{array}\right\} = 0.
$$

[Wurzeln dieser Gleichung vom 24. Grade sind $\sin \operatorname{lemn} \frac{k\varpi}{5}$ für $k = 1, 2, 3, 4$, die übrigen sind imaginär, also:]

Determ. rad. imag.

$$\frac{\square}{\square} = \frac{4(1-x^4)}{\square},$$

[setze $x^4 = y$, so ist]

$$(3 - 6y - yy)(1 + y) = 2(1 + 6y - 3yy)\sqrt{(1 - y)}$$

$$S = \sqrt{720} - 26 \; [= 12\sqrt{5} - 26]$$

$$\left[= \left(\sin \operatorname{lemn} \frac{2\varpi}{5}\right)^4 + \left(\sin \operatorname{lemn} \frac{4\varpi}{5}\right)^4\right]$$

$$\begin{array}{r} 349 - 156\sqrt{5} \\ -9 + 4\sqrt{5} \\ \hline 340 - 152\sqrt{5} \end{array} \left[= \tfrac{1}{2}\left(\sin \operatorname{lemn} \frac{4\varpi}{5}\right)^4 - \tfrac{1}{2}\left(\sin \operatorname{lemn} \frac{2\varpi}{5}\right)^4\right]^2.$$

Zwei Wurzeln obiger Gleichung sind

$$+\,0{,}0733810047 \quad \left[= \left(\sin \operatorname{lemn} \frac{2\varpi}{5}\right)^4\right]$$

$$+\,0{,}7594355 \quad \left[= \left(\sin \operatorname{lemn} \frac{4\varpi}{5}\right)^4\right].$$

[5.]

[S. 100—101]

Auflösung der Gleichung

$$5 - 62x - 105xx + 300x^3 - 125x^4 + 50x^5 + x^6 \; [= 0\,^*)]$$

[Es folgt eine Zahlenrechnung, anscheinend nach der Regula falsi].

Also eine Wurzel

$$= 0{,}07338100477 \quad \left[= \left(\sin \operatorname{lemn} \frac{2\varpi}{5}\right)^4\right]$$

und folglich

$$\sin 36^0 = 0{,}52047024 \; [= \sqrt[4]{0{,}073381}].$$

[*) Das x in dieser Gleichung ist die vierte Potenz der im art. [4.] ebenso bezeichneten Unbekannten; die Gleichung hat also die Wurzeln $\left(\sin \operatorname{lemn} \frac{k\varpi}{5}\right)^4$ für $k = 1, 2, 3, 4$.]

Hieraus aber folgt eine zweite Wurzel

$$= 16\,\frac{(1-t)^2 t}{(1+t)^2}$$

$$\begin{aligned} l16 &= 1{,}2041200 \\ lt &= 0{,}8655836-2 \quad [= \log_{10} 0{,}073381\ldots = \log_{10}(\text{sin lemn } 36^0)^4] \\ l(1-t)^2 &= 0{,}9338025-1 \\ -l(1+t)^2 &= 0{,}8769844-1 \\ \hline &\quad 0{,}8804905-1 \quad [= \log_{10}]\ 0{,}7594348 \quad [= \log_{10}(\text{sin lemn } 72^0)^4]. \end{aligned}$$

Also

$$l \sin 72^0 = 0{,}9701226$$

und

$$\sin 72^0 = 0{,}9335179.$$

[6.]

[Ein Zettel Fh Nr. 1, Kapsel 50.]

Rechnungen zur Lemniscata gehörig.

$$(\sin 0{,}4)^4 + (\sin 0{,}8)^4 = 12\sqrt{5} - 26\,[^*)]$$
$$= 2(0{,}4164078649 \quad 9873817845 \quad 5042012387 \quad 65741)$$

$$(\tfrac{1}{2}(\sin 0{,}8)^4 - \tfrac{1}{2}(\sin 0{,}4)^4)^2 = 340 - 152\sqrt{5}$$
$$= 0{,}1176674200 \quad 3196614580 \quad 5602352846 \quad 01228.$$

Daraus Radix

$$\begin{aligned} &0{,}3430268503 \quad 0761971797 \quad 7310507555 \quad 85731 \\ &\qquad\qquad\qquad\qquad\qquad\qquad -0463415326 \quad 98758 \\ \hline &\qquad\qquad\qquad\qquad\qquad\qquad\ \ 6847092228 \quad 86973 \end{aligned}$$

Also

$$(\sin 0{,}4)^4 = 0{,}0733810146 \quad 9111846047 \quad 7731504831 \quad 80010$$

Daraus Radix

$$(\sin 0{,}4)^2 = 0{,}2708893033 \quad 8999814497 \quad 30710$$

[*) sin 0,4 und sin 0,8 bedeuten $\sin\frac{4\varpi}{10}$ und $\sin\frac{8\varpi}{10}$, d. h. also sin lemn $\frac{2\varpi}{5}$ und sin lemn $\frac{4\varpi}{5}$.]

21*

Observatio per inductionem facta gravissima theoriam residuorum biquadraticorum cum functionibus lemniscatis elegantissime nectens. Puta si a+bi est numerus primus, a-1+bi per 2+2i divisibilis, multitudo omnium solutionum congruentiae

$$1 \equiv xx + yy + xxyy \pmod{a + bi}$$

inclusis

$$x = \infty, \quad y = \pm i; \qquad x = \pm i, \quad y = \infty,$$

fit

$$= (a-1)^2 + bb.$$

We have progressed a little beyond this last entry in the intervening 185 years, but not so much as one might think! Without going into further detail, which would be too much of a digression, I observe that if

$$1 = x^2 + y^2 + x^2y^2$$

and

$$z = y(1 + x^2)$$

then

$$z^2 = y^2(1+2x^2+x^4) = y^2(1+x^2)+y^2x^2(1+x^2) = 1-x^2+x^2(1-x^2) = 1-x^4.$$