Isoperimetric Inequality and higher order curvatures

Yi Wang

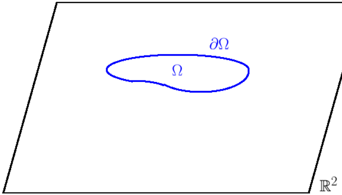
Institute for Advanced Study, Oct 1, 2014

Classical isop inequalities

Classical isop inequality on a planar domain:

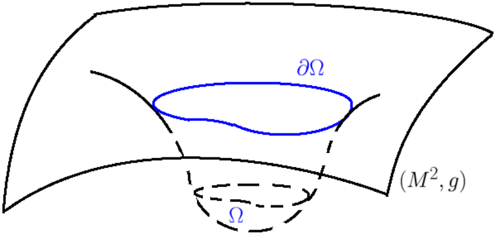
Suppose $\Omega \subset \mathbb{R}^2$ is a bounded domain, and $\partial\Omega$ is its smooth boundary. Then

$$Area(\Omega) \leq \frac{1}{4\pi} Perimeter(\partial\Omega)^2. \quad (1)$$



Classical isop inequalities

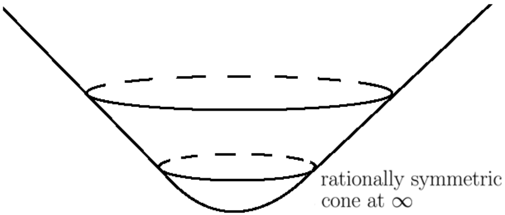
What about a curved surface?



The same *Perimeter*$(\partial\Omega)$ bounds a domain Ω with bigger area.

Classical isop inequalities

If $\int_{M^2} K_g^+(x) dv_g(x)$ is not too big, then the isop inequality is still valid.



For instance, on a cone, the isop inequality is valid and $\int_{M^2} K_g^+(x) dv_g(x) < 2\pi$.

Classical isop inequalities

Isop inequality on a curved surface (Fiala-Huber '1940-50's)

(M^2, g) is a noncompact complete simply connected surface, and $\Omega \subset M^2$. If $\int_{M^2} K_g^+(x) dv_g(x) < 2\pi$, then

$$Area(\Omega) \leq \frac{1}{2(2\pi - \int_{M^2} K_g^+(x) dv_g(x))} Perimeter(\partial\Omega)^2. \tag{2}$$

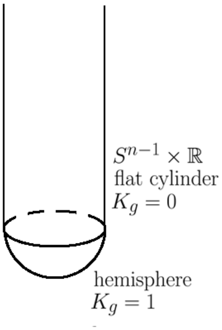
Here $K_g(x)$ denotes the Gaussian curvature.
$K_g^+(x) \overset{def}{=} \max\{K_g(x), 0\}$.

For $\mathbb{R}^2$, $K_g(x) \equiv 0$. Thus (2) covers (1) on a planar domain.

Classical isop inequalities

The bound 2π in $\int_{M^2} K_g^+(x) dv_g(x) < 2\pi$ is sharp:

Counter example: $M^2 =$ Half Cylinder. $\int_{M^2} K_g^+(x) dv_g(x) = 2\pi$. It does not satisfy the isop inequality.

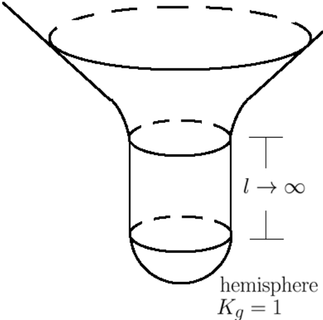


Isop inequality on a curved surface (P. Li and L. Tam '91)

(M^2, g) is a noncompact complete simply connected surface, and $\Omega \subset M^2$. If K_g is integrable and $\int_{M^2} K_g(x) dv_g(x) < 2\pi$, then

$$Area(\Omega) \leq C(g) Perimeter(\partial\Omega)^2. \tag{3}$$

Can C be independent of g? No. Because:



Questions in higher dimensions

Known: Isop inequality under POINTWISE curvature assumptions:

- Aubin, Cantor '70's
- Coulhon '89 and Saloff-Coste, Varopoulos '93

Question 1

Can we prove isop inequality by assuming INTEGRAL condition instead of POINTWISE condition?

In other words, curvature may be ∞ on some points and the isop inequality is still valid.
For instance, the rotationally symmetric cone only has ∞ curvature at the vertex, it satisfies isoperimetric inequality.

Questions in higher dimensions

Question 2:

What is the suitable curvature?

Many possible choices: $Riem_g$, Rc_g, R_g, etc.
Notice: on surfaces, they are all equal to the Gaussian curvature K_g.

Question 3:

Is conformal structure relevant to the isop inequality in higher dimensions?

Q-curvature

So far, we can answer the questions in the conformally flat case. The isop behavior is controlled by the integrals of Q-curvature.

Definition

In dimension 4,

$$Q_g \stackrel{def}{=} \frac{1}{12}\left\{-\Delta R + \frac{1}{4}R^2 - 3|E|^2\right\},$$

E is traceless part of Ric. In other dimensions, it has other expressions.

Q-curvature

Analogy between Q_g and K_g

- Q-curvature has analogous transformation law as that of the Gaussian curvature.
- Q-curvature appears in the Chern-Gauss-Bonnet formular.

First result

Li-Tam's analogue in higher dimension holds.

Theorem

(W. '11) Suppose $(M^n, g) = (\mathbb{R}^n, e^{2u}|dx|^2)$ *is a noncompact complete Riemannian manifold with* $\lim_{|x|\to\infty} R_g \geq 0$. *If its Q-curvature satisfies*

$$\int_{M^n} |Q_g| dv_g < \infty, \quad \int_{M^n} Q_g dv_g < c_n, \tag{4}$$

then the manifold satisfies the isop inequality:

$$|\Omega|_g \leq C(g)|\partial\Omega|_g^{n/(n-1)}. \tag{5}$$

Here c_n *is equal to the integral of the Q-curvature on a hemi-sphere.* $c_2 = 2\pi$, $c_4 = 4\pi^2$, *etc.*

Results and questions

Fiala-Huber's analogue in higher dimension holds. c_n is sharp.

Theorem

(W. '13) Suppose $(M^n, g) = (\mathbb{R}^n, e^{2u}|dx|^2)$ *is a noncompact complete Riemannian manifold with* $\lim_{|x|\to\infty} R_g \geq 0$. *If its Q-curvature satisfies*

$$\alpha \overset{def}{=} \int_{M^n} Q_g^+ dv_g < c_n \tag{6}$$

and

$$\beta \overset{def}{=} \int_{M^n} Q_g^- dv_g < \infty, \tag{7}$$

then the manifold satisfies the isop inequality:

$$|\Omega|_g \leq C(\alpha, \beta, n)|\partial\Omega|_g^{n/(n-1)}. \tag{8}$$

Results and questions

When c_n is replaced by ϵ, the result was known before by M. Bonk, J. Heinonen and E. Saksman '08. This assumption is pushed to the sharp ones in the above theorems.

Results and questions

What happens in non-conformally flat case? How does $\int |W|^n/2$ play a role in the problem?

The end

Thank you!