De Rham Complexes and Derived Intersections

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is *formal* if it is quasi-isomorphic to

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If $\mathscr{F}^\bullet$ is a dg algebra (has an algebra structure), want quasi-isomorphism to respect this.

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- de Rham complexes in characteristic $p > 0$ (Deligne-Illusie 1988, Ogus-Vologodsky 2005)
- derived intersections of subvarieties (Arinkin-Căldăraru 2010, Arinkin-Căldăraru-Hablicsek 2014)

M. Mustață noticed a similarity between these two results, asked if one can be phrased in terms of the other.

Algebraic de Rham cohomology

(Grothendieck): Algebraic de Rham complex:

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Theorem (Grothendieck). If X is smooth, projective over $\mathbb{C}$,

$$H^*_{\mathrm{dR}}(X) \cong H^*(X^{\mathrm{an}}, \mathbb{C}).$$

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Over $\mathbb{C}$, the Hodge theorem tells us this spectral sequence degenerates at ^1E.

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Theorem (Cartier). $F_* \Omega^\bullet_{X,\mathrm{dR}}$ *is a complex of coherent* $\mathscr{O}_{X'}$-modules. Its cohomology is

$$\mathscr{H}^i(F_* \Omega^\bullet_{X,\mathrm{dR}}) \cong \Omega^i_{X'}.$$

Theorem (Deligne-Illusie). Assume $p > \dim X$. If X admits a flat lift to $W_2(k)$, then the complex $F_* \Omega^\bullet_{X,\mathrm{dR}}$ is *formal*, i.e.,

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From this Deligne-Illusie are able to get an algebraic proof of the Hodge theorem in characteristic zero.

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- $\Omega^\bullet_{X,\mathrm{dR}} : 0 \to k[x] \xrightarrow{d} k[x]dx \to 0$;
- Deligne-Illusie + Cartier:

$$\begin{array}{lccccccccc}
F_*\Omega^\bullet_{X,\mathrm{dR}} : & 0 & \longrightarrow & k[x] & \xrightarrow{d} & k[x]dx & \longrightarrow & 0 \\
& & & \uparrow \cup & & \uparrow C & & \\
\bigoplus_i \Omega^i_{X'}[-i] : & 0 & \longrightarrow & k[x^p] & \xrightarrow{0} & k[x^p]d(x^p) & \longrightarrow & 0
\end{array}$$

C sends $d(x^p)$ to $x^{p-1}dx$.

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Think of it as an infinitesimal thickening of W^0.

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Think of $\mathscr{O}_W$ as $\mathrm{Sym}_{W^0}(N^\vee_{X/S}[1])$.
So $W = \mathrm{Tot}_{W^0} N_{X/S}[-1]$.

Theorem (Arinkin-Căldăraru-Hablicsek). The cohomology sheaves of $\mathscr{O}_W$ are

$$\mathscr{H}^{-i}(\mathscr{O}_W) \cong \wedge^i E^\vee,$$

where

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Assume $p = 0$ or $p > \dim S$. If, moreover, the short exact sequence

$$0 \to TX + TY \to TS \to E \to 0$$

splits, then $\mathscr{O}_W$ is formal:

$$\mathscr{O}_W \cong \bigoplus_i \wedge^i E^\vee[i] = \mathrm{Sym}_{W^0}(E^\vee[1])$$

so

$$W \cong \mathrm{Tot}_{W^0} E[-1].$$

Similarity with Deligne-Illusie

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This is similar to the condition that X should admit a flat lift to $W_2(k)$ in the Deligne-Illusie theorem. ($W_2(\mathbb{Z}/p\mathbb{Z}) = \mathbb{Z}/p^2\mathbb{Z}$ is the first infinitesimal neighborhood of $\operatorname{Spec} \mathbb{Z}/p\mathbb{Z}$ in $\operatorname{Spec} \mathbb{Z}$.)

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Question (Mircea Mustaţă). Can one recast the computation of $F_*\Omega^\bullet_{X,\mathrm{dR}}$ in Deligne-Illusie as a derived intersection problem, to deduce its formality from Arinkin-Căldăraru?

Azumaya spaces

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Definition. An Azumaya space $\overline{S}$ is a pair $(S, \mathscr{A})$ consisting of a space (scheme, dg scheme) X together with a sheaf $\mathscr{A}$ of Azumaya algebras on S. A morphism $(S, \mathscr{A}) \to (T, \mathscr{B})$ of Azumaya spaces is a morphism $f : S \to T$ of spaces, together with a Morita equivalence between $f^*\mathscr{B}$ and $\mathscr{A}$:

$$\mathfrak{Mod}(f^*\mathscr{B}) \xrightarrow{\sim} \mathfrak{Mod}(\mathscr{A}).$$

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Example. $(S, \mathscr{A})$ an Azumaya space, X a subvariety of S such that $\mathscr{A}|_X$ splits ($\mathscr{A}|_X \cong \mathrm{End}_X(E)$ for some vector bundle E on X). Get a morphism

$$(X, \mathscr{O}_X) \to (S, \mathscr{A}).$$

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- Splittings compatible on the underived intersection W^0:

$$E_X|_{W_0} \cong E_Y|_{W_0}.$$

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If both of them are formal (Arinkin-Căldăraru-Hablicsek theorem) we conclude that $\mathscr{O}_W \cong \mathscr{O}_{\overline{W}}$.

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Then $Z(D) = k[x^p, \partial^p] = \mathscr{O}_{T^*X'}$ and

$$D|_{X'} = k\{x, \partial\}/(\partial^p) = \mathrm{End}_{k[x^p]}(k[x]).$$

Deligne-Illusie vs. Arinkin-Căldăraru

Theorem. Let $S = T^*X'$, $\overline{S} = (S, D)$. D splits on the subvariety X' of S, so we can consider the twisted and untwisted derived intersections

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Applying the theorem comparing $\mathscr{O}_W$ and $\mathscr{O}_{\overline{W}}$ yields the Deligne-Illusie theorem.

Twisted de Rham cohomology

Definition. Assume X quasi-projective, and fix a regular function f on X.
The *twisted de Rham complex* is

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Analogue of the Hodge-de Rham degeneration for matrix factorizations.

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We can prove this isomorphism using new techniques. However, we need to use an elementary-looking statement from Ogus-Vologodsky for which we do not know any easy proof.

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independent of choices of coordinates, where I is the ideal given locally by $(f_1^p, \ldots, f_n^p)$, $f_1, \ldots, f_n$ the partial derivatives of f.

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independent of choices of coordinates, where I is the ideal given locally by $(f_1^p, \ldots, f_n^p)$, $f_1, \ldots, f_n$ the partial derivatives of f.

(This is the problem of showing that $E_X|_{W^0} \cong E_Y|_{W^0}$ in the twisted derived intersection problem.)

An elementary problem

Let f be a regular function on a smooth quasi-projective variety X, over a field k of characteristic $p > 0$.

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We know this is true using a difficult result of Ogus-Vologodsky.

Conjecture. A solution (over affine space) can be given as

$$g = \prod_i \operatorname{hexp}(m_i)$$

where

$$f = \sum_i m_i$$

is a decomposition of f into monomials, in some coordinates, and

$$\operatorname{hexp}(x) = \exp\left(x + \frac{x^p}{p} + \frac{x^{p^2}}{p^2} + \cdots\right)$$

is the Artin-Hasse exponential.

Thank you!