Cubic fourfolds and K3 surfaces

Joint work with Nick Addington

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At the end of the talk we show these loci are (almost) the same.

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- cf. H^2(K3 surface S) 1 20 1 Signature (3,19)
- Generator $\sigma^{2,0} \in H^{2,0}(S)$ defines period point in $H^2(S,\mathbb{C})$
  $\Rightarrow$ Torelli theorem (Pjateckiĭ-Šapiro–Šafarevič, Burns–Rapoport).

Not same unless pass to codimension-1 sub-Hodge structure of signature $(2,19)$ in both cases.

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- Moduli space of *special* cubics fourfolds of discriminant d.

Hassett's theorem

Identifies precisely when the orthogonal lattices

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and

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Theorem (Hassett)

Fix a special cubic fourfold (X, T) of discriminant $d = \mathrm{disc}\,\langle h^2, T\rangle$. There exists a polarised K3 surface (S, L) such that

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$(*)$ *d even, not divisible by 4,9, nor any prime $6n+5$.*

That is $d = (6), 14, 26, 38, \ldots$. This is then $\deg(L)$ also.

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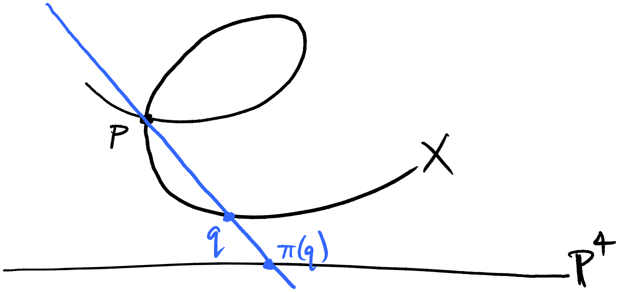
- Gives birational map $\pi \colon X \dashrightarrow \mathbb{P}^4, \quad q \mapsto L$



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(The correspondence in $X \times S$ actually gives a Fourier-Mukai kernel in $D(X \times S)$ yielding $D(S) \hookrightarrow D(X)$ – see later.)

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This gives a correspondence $\subset X \times S$ (and FM kernel in $D(X \times S)$) giving

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Pfaffian cubics are also all rational.

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$\Rightarrow \mathbb{P}^2$ family of quadric surfaces Q; in fact

$$\mathrm{Bl}_P X \to \mathbb{P}^2$$

is a quadric surface fibration, generic fibre $\mathbb{P}^1 \times \mathbb{P}^1$, singular fibres (cone over a conic) over discriminant sextic curve $\subset \mathbb{P}^2$.

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When $\mathrm{Br} \neq 0$, $H^2_{\mathrm{prim}}(S, \mathbb{Z}) \not\to H^4_{\mathrm{prim}}(X, \mathbb{Z})$ (unless work over $\mathbb{Z}[\frac{1}{2}]$ or $\mathbb{Q}$).

Non-example $d = 8$ continued (continued)

If there exists *another* class $T' \in H^{2,2}(X, \mathbb{Z})$ (as well as P and h^2) such that $\int_Q T' = 1$

(since $\int_X h^2 \wedge (h^2 - P) = 2$ sufficient to ask that $\int_X T' \wedge (h^2 - P)$ is *odd*)

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But $d = 8$ is not on the list $(*)$?

Example $d = 8$ and $d \in (*)$

In fact $d = 8$ and $\mathrm{Br} = 0$ ($\iff \exists T'$ with $T'.(h^2 - P) = 1$) $\iff$

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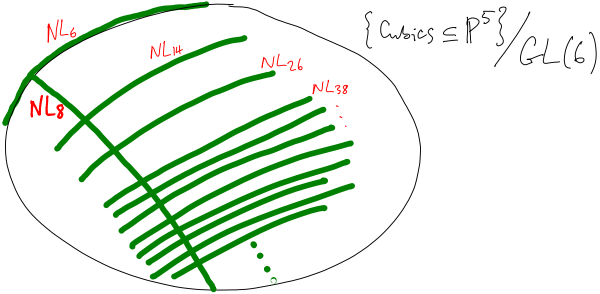
That is $X \in NL_8 \cap NL_d$.
And all NL_d intersect NL_8 for d satisfying $(*)$.



And now we have $H^2_{\mathrm{prim}}(S, \mathbb{Z}) \hookrightarrow H^4_{\mathrm{prim}}(X, \mathbb{Z})$ and rationality.

Rationality conjecture

Harris and Hassett (cautiously) asked whether X might be rational if and only if

$$\langle h^2, T\rangle^\perp \;\cong\; H^2_{\mathrm{prim}}(S,\mathbb{Z})$$

for some polarised K3 surface (S, L) and class

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There is one thing better than correspondences:
Fourier-Mukai kernels.

Kuznetsov categorifies Hassett's approach, in some sense.

Kuznetsov's approach through derived categories

$$D(X) \;=\; \big\langle \mathcal{A}_X, \mathcal{O}_X, \mathcal{O}_X(1), \mathcal{O}_X(2) \big\rangle,$$

where

$$\begin{aligned} \mathcal{A}_X \;&:=\; \langle \mathcal{O}_X, \mathcal{O}_X(1), \mathcal{O}_X(2) \rangle^{\perp} \\ &=\; \big\{ E \in D(X) \colon RHom(\mathcal{O}_X(i), E) = 0 \text{ for } i = 0, 1, 2 \big\}. \end{aligned}$$

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$\mathcal{O}_X, \mathcal{O}_X(1), \mathcal{O}_X(2)$ form an *exceptional collection* so can use Gram-Schmidt to project any $E \in D(X)$ into $\mathcal{A}_X$.
(Replace E by cone of $RHom(\mathcal{O}(i), E) \otimes \mathcal{O}(i) \to E$, etc.)

$$\mathcal{A}_X \underset{\pi_{\mathcal{A}}}{\overset{\hookrightarrow}{\leftarrows}} D(X)$$

$\mathcal{A}_X$ is a noncommutative K3 surface

$\mathcal{A}_X$ is a 2-dimensional Calabi-Yau category (it has Serre functor [2])

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Kuznetsov shows that the known rational cubics X indeed have $\mathcal{A}_X$ geometric, i.e. $D(S) \hookrightarrow D(X)$.

Noone has yet proved a single cubic X to be irrational.
(But: Francois Greer and Jun Li ?)

$d = 8$ again

Recall $P \subset X$, and the fibrations

$$\begin{array}{ccc} Q \overset{\iota}{\hookrightarrow} \mathrm{Bl}_P X & \qquad & S = \{\text{rulings of fibres}\} \\ \downarrow & & \downarrow\!\!\downarrow \\ \mathbb{P}^2 & & \mathbb{P}^2 \end{array}$$

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So X geometric if $Br = 0$, which we saw meant $X \in NL_8 \cap NL_d$ for some $d \in (*)$.

Hassett = Kuznetsov ?

We would like to show that the two rationality conjectures are the same. That is,

$$X \in NL_d \;\text{ for } d \text{ satisfying } (*) \iff \mathcal{A}_X \text{ geometric},$$

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Expect loci are equal, but taking closure of above result tricky. (Limits of FM kernels.)

Algebraic cycles

Taking limits of algebraic cycles is easy, however.
(The Hilbert scheme is proper.)

Corollary

Given any X in Hassett's locus, his Hodge isometry

$$H^2_{\mathrm{prim}}(S,\mathbb{Z})(-1) \longrightarrow \langle h^2, T\rangle \subset H^4_{\mathrm{prim}}(X,\mathbb{Z})$$

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We can strengthen this slightly.

Theorem

Fix any cubic X and K3 surface S. If a Hodge class $Z \in H^{3,3}(S\times X,\mathbb{Q})$ induces a Hodge isometry of integral transcendental lattices

$$T(S)(-1) \xrightarrow{\sim} T(X)$$

then Z is algebraic.

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- The resulting equivalence $\mathcal{A}_X \cong D(\mathcal{M})$ is the right one for NL_d! (It expresses $\mathcal{M}$ as a moduli space of objects of type a, and a deforms along NL_d.)

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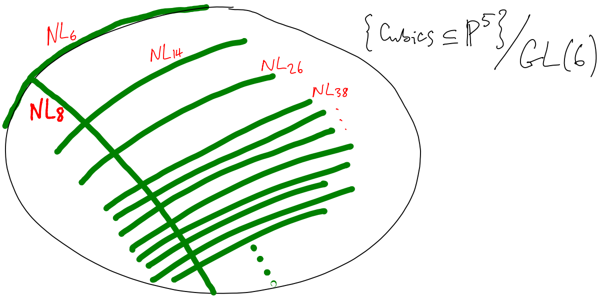

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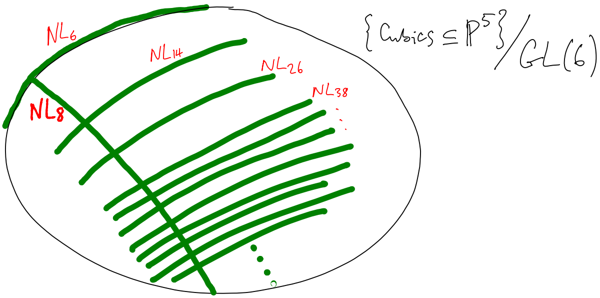

- Finally deform X into NL_d from $NL_d \cap NL_8$, and deform $\mathcal{M}$ with it (as an abstract K3, via Hassett's result and Torelli).
- Need to show the FM kernel $U \in D(\mathcal{M} \times X)$ deforms to all orders. (Since NL_d irreducible this shows it deforms to a dense Zariski open. The FM functor being full and faithful is also an open condition.)

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Identify this obstruction with

$$\kappa_{\mathcal{M}} - \kappa_X.$$

$\kappa_{\mathcal{M}} \in H^{1,1}(\mathcal{M})$ is the Kodaira-Spencer class of the deformation of $\mathcal{M}$ (contracted with $\sigma^{2,0}_{\mathcal{M}}$), and $\kappa_X \in H^{2,2}(X) \supset H^{1,1}(M)$ is the same for X.

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There is a Hassett/Addington cohomological condition for this too:

$$(**) \qquad d = \frac{2n^2+2n+2}{a^2} \text{ for some } n, a \in \mathbb{Z}.$$

And $(**) \Rightarrow (*)$ but $(*) \not\Rightarrow (**)$.

In particular, the derived category would then having nothing to do with rationality.