IAS, Semester on Non-equilibrium dynamics and RMT

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All order asymptotics for β-ensembles in the multi-cut regime

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All order asymptotics for β-ensembles in the multi-cut regime

1. Beta-ensembles and random matrices

2. Applications to orthogonal polynomials

3. Ideas about the proof

4. Perspectives

The 1d beta-ensemble is ...

- ... the probability measure on $A^N \subseteq \mathbb{R}^N$

$$\mathrm{d}\mu_N^A = \frac{1}{Z_N^A} \prod_{1 \leq i < j \leq N} |\lambda_i - \lambda_j|^\beta \prod_{i=1}^N e^{-N(\beta/2)V(\lambda_i)} \mathbf{1}_A(\lambda_i)\mathrm{d}\lambda_i \qquad \beta > 0$$

- It is the measure induced on eigenvalues of a random matrix M

$\mathrm{d}M\, e^{-N(\beta/2)\,\mathrm{Tr}\,V(M)}$

$\beta = 1$ real symmetric matrices

$\beta = 2$ hermitian matrices

$\beta = 4$ quaternionic self-dual matrices

Wigner, Dyson, Mehta (50s-60s)

M = triagonal — all $\beta > 0\,, V$ quadratic

Dumitriu, Edelman '02

We would like to study when N → ∞ ...

- the (random) empirical measure

$$L_N = \frac{1}{N} \sum_{i=1}^{N} \delta_{\lambda_i}$$

Example : what kind of random variable is $\sum_{i=1}^{N} f(\lambda_i)$?

- the partition function

$$Z_N^A = \int_{A^N} \prod_{1 \le i < j \le N} |\lambda_i - \lambda_j|^\beta \prod_{i=1}^{N} e^{-N(\beta/2)V(\lambda_i)} \mathrm{d}\lambda_i$$

$$L_N = \frac{1}{N}\sum_{i=1}^{N}\delta_{\lambda_i} \qquad Z_N^A = \int_{A^N}\prod_{1\le i<j\le N}|\lambda_i-\lambda_j|^\beta \prod_{i=1}^{N} e^{-N(\beta/2)V(\lambda_i)}\mathrm{d}\lambda_i$$

The leading order ... is given by a continuous approximation

Classical result

Mhaskar, Saff '85 Anderson, Guionnet, Zeitouni (book '09) Boutet de Monvel, Pastur, Shcherbina '95

Assume V continuous and confining $\left(\liminf_{|x|\to\infty}\frac{V(x)}{2\ln|x|}>1\right)$

- $\mathcal{E}[\mu] = \iint \ln|x-y|\mathrm{d}\mu(x)\mathrm{d}\mu(y) - \int V(x)\mathrm{d}\mu(x)$
  has a unique maximizer $\mu_{\mathrm{eq}} \in \mathcal{M}^1(A)$ characterized by
  $\exists C \quad 2\int_A \ln|x-y|\mathrm{d}\mu_{\mathrm{eq}}(y) - V(x) \le C$ with equality μ_{eq} everywhere
- $L_N \longrightarrow \mu_{\mathrm{eq}}$ almost surely and in expectation
- $Z_N^A = \exp\left\{N^2(\beta/2)(\mathcal{E}[\mu_{\mathrm{eq}}]+o(1))\right\}$

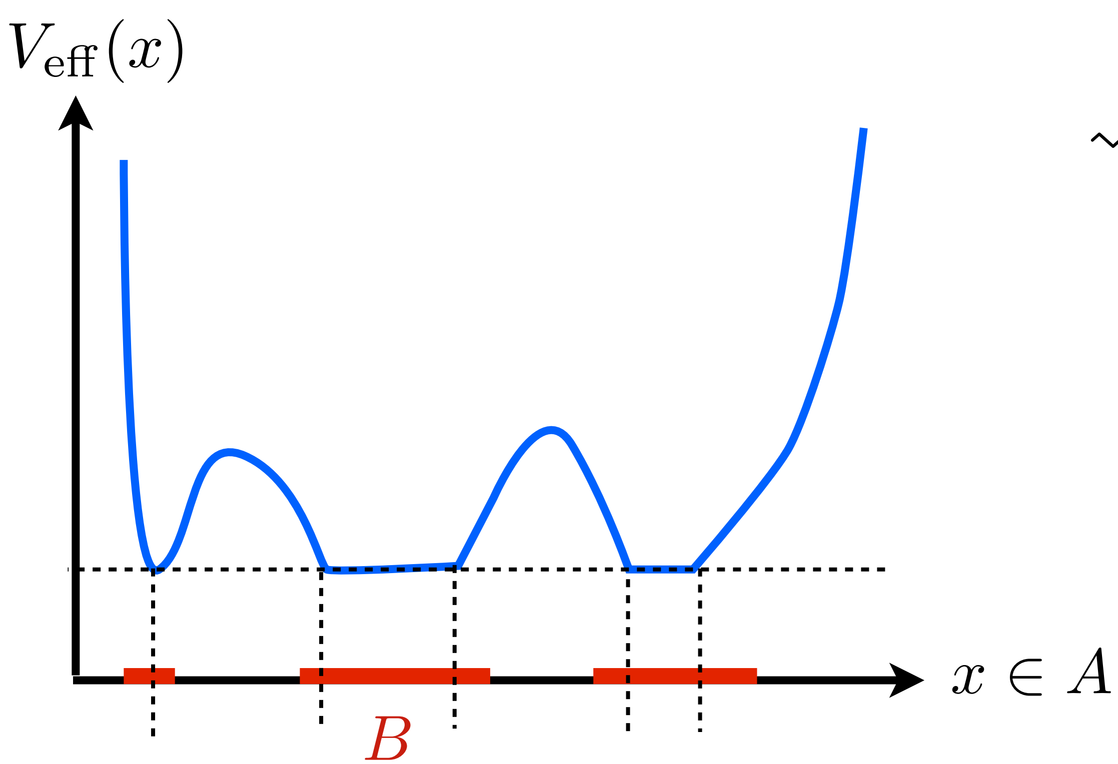
Large deviations (local result)

- λ_i's feel the effective potential

$$V_{\text{eff}}(x) = V(x) - 2\int \ln|x-y|\mathrm{d}\mu_{\text{eq}}(y) - C \geq 0$$

- For any closed $F \subseteq A$

$$\text{Prob}_N^A\big[\exists i\ \ \lambda_i \in F\big] \leq \exp\Big(- N(\beta/2)\big\{\min_{x\in F} V_{\text{eff}}(x) + o(1)\big\}\Big)$$



$\rightsquigarrow$ One can restrict to a compact $B \subseteq A$ neighborhood of $\{V_{\text{eff}}(x) = 0\}$

$$Z_N^B = Z_N^A(1 + o(e^{-cN}))$$

Large deviations (global result)

- $\mathfrak{D}[\mu_1,\mu_2] = -\int \ln|x-y|\mathrm{d}(\mu_1-\mu_2)(x)\mathrm{d}(\mu_1-\mu_2)(y) = \int_0^{\infty} \frac{|\mathrm{FT}[\mu_1-\mu_2](k)|^2}{k}$

  defines a distance $\in [0,+\infty]$ on $\mathcal{M}^1(A)$

  such that $\left|\int f(x)\mathrm{d}(\mu_1-\mu_2)(x)\right| \leq \sqrt{2}\Big(\int_{\mathbb{R}} k\,|\mathrm{FT}[f](k)|^2\mathrm{d}k\Big)^{1/2}\mathfrak{D}^{1/2}[\mu_1,\mu_2]$

- Let us pick a nice regularization $L_N = \frac{1}{N}\sum_{i=1}^{N}\delta_{\lambda_i} \rightsquigarrow \widetilde{L}_N$

Proposition **Borot, Guionnet ('11)**

If V is $\mathcal{C}^3$, we have for N large enough

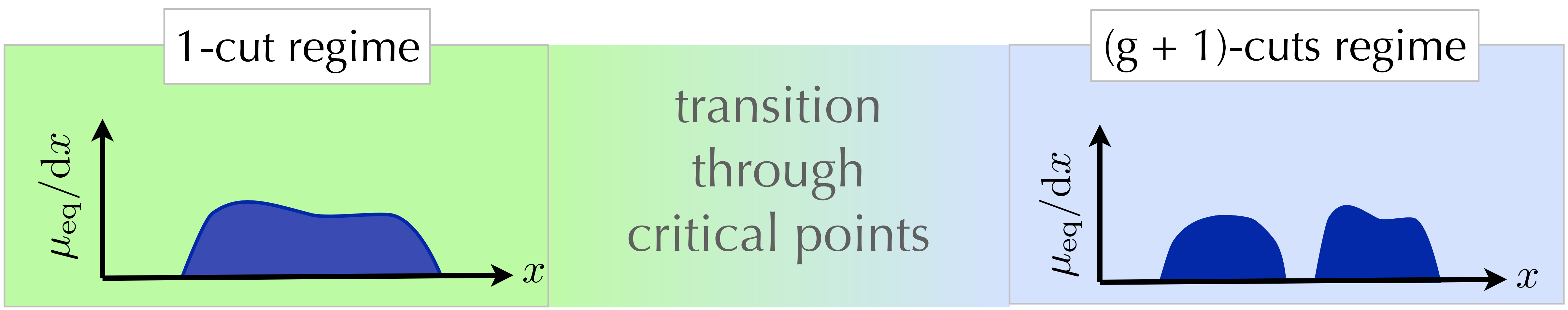
$$\mathrm{Prob}_N^A\big[\mathfrak{D}^{1/2}[\widetilde{L}_N,\mu_{\mathrm{eq}}] \geq t\big] \leq \exp\big(CN\ln N - N^2(\beta/2)t^2\big)$$

More on the equilibrium measure ...

- V real-analytic $\Longrightarrow \begin{cases} \mu_{\mathrm{eq}} \text{ is supported on a finite number of segments} \\ S = \bigcup_{h=0}^{g} [a_h, b_h] \end{cases}$

- $\alpha \in \partial S$ is a hard edge if $\alpha \in \partial A$, is a soft edge otherwise

$$\mathrm{d}\mu_{\mathrm{eq}}(x) = \frac{\mathbf{1}_S(x)\mathrm{d}x}{2\pi} M(x) \prod_{\alpha \text{ soft}} |x-\alpha|^{1/2} \prod_{\alpha \text{ hard}} |x-\alpha|^{-1/2}$$



- We say that μ_{eq} is off-critical when $M(x) > 0$ on A

Finite size corrections : we assume ...

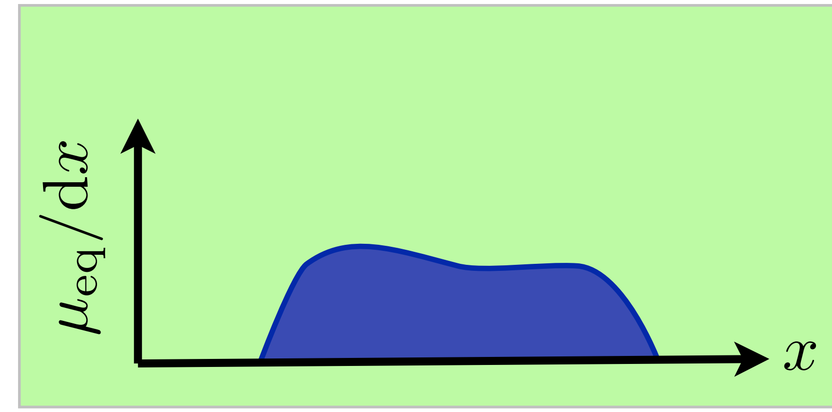
- $V = V_0 + (1/N)V_1 + \cdots$ $\begin{cases} V_0 & \text{real analytic on } A \\ V_1 & \text{complex analytic on } A \end{cases}$

- Control of large deviations $V_{\mathrm{eff}}(x) > 0$ for $x \in A \setminus S$

- μ_{eq} is off-critical

- f = test function, analytic on A

Result in the 1-cut regime



- 1/N asymptotic expansion

$$Z_N^A = N^{\gamma N + \gamma'} \exp\Big[\sum_{k \geq -2} N^{-k} F_k + O(N^{-\infty})\Big]$$

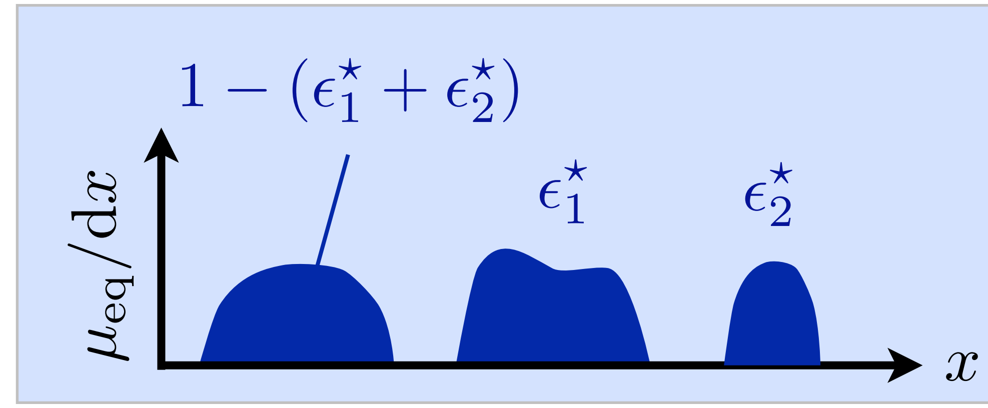
γ, γ' depend only on β and the nature of the edges

$$F_k = \sum_{h=0}^{\lfloor k/2 \rfloor + 1} \Big(\frac{\beta}{2}\Big)^{1-h} \Big(1 - \frac{2}{\beta}\Big)^{k+2-2h} F_{[h];k+2-2h}$$

- Central limit theorem

$$\Big(\sum_{i=1}^{N} f(\lambda_i) - N \int_A f(\xi)\mathrm{d}\mu_{\mathrm{eq}}(\xi)\Big) \longrightarrow \text{ (non-centered) gaussian}$$

Result in the (g + 1)-cuts regime



- Oscillatory asymptotic expansion

$$Z_N^A = N^{\gamma N+\gamma'}(\mathcal{D}_N\Theta_{-N\epsilon_\star})\big(F'_{-1}\big|F''_{-2}\big)\exp\Big[\sum_{k\geq -2} N^{-k}\,F_k + O(N^{-\infty})\Big]$$

where $\mathcal{D}_N = \sum_{r\geq 0}\frac{1}{r!}\sum_{\substack{\ell_1,\ldots,\ell_r\geq 1\\ k_1,\ldots,k_r\geq -2\\ \sum_i(k_i+\ell_i)>0}} N^{-(\sum_i k_i+\ell_i)}\prod_{i=1}^{r}\frac{F_{k_i}^{(\ell_i)}\cdot\nabla_{\mathbf{w}}^{\otimes \ell_i}}{\ell_i!}$

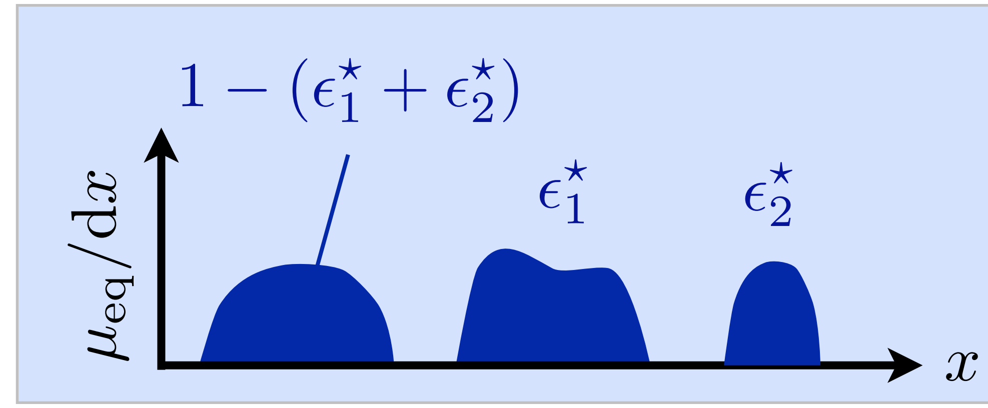
acts as a differential operator on the Siegel theta function

$$\Theta_\mu(\mathbf{w}|\mathbf{Q}) = \sum_{\mathbf{m}\in\mathbb{Z}^g} e^{\mathbf{w}\cdot(\mathbf{m}+\mu)+\frac{1}{2}(\mathbf{m}+\mu)\cdot\mathbf{Q}\cdot(\mathbf{m}+\mu)}$$

- Moving characteristics $\mu = -N\epsilon_\star \bmod \mathbb{Z}^g$

  Quadratic form $\mathbf{Q} = F''_{-2} = 2\mathrm{i}\pi\,(\beta/2)\times(\text{period matrix}) < 0$

Result in the (g + 1)-cuts regime



- No central limit theorem in general ...

$$\mathbb{E}\Big[e^{\mathrm{is}\left(\sum_{i=1}^N f(\lambda_i)-N\int f(x)\mathrm{d}\mu_{\mathrm{eq}}(x)\right)}\Big] \underset{N\infty}{\sim} e^{\mathrm{is}\,m_1[f]-m_2[f]s^2/2}\,\frac{\Theta_{-N\epsilon^\star}\left(F'_{-1}+\mathrm{is}\,v[f]|F''_{-2}\right)}{\Theta_{-N\epsilon^\star}\left(F'_{-1}|F''_{-2}\right)}$$

(non-centered) gaussian

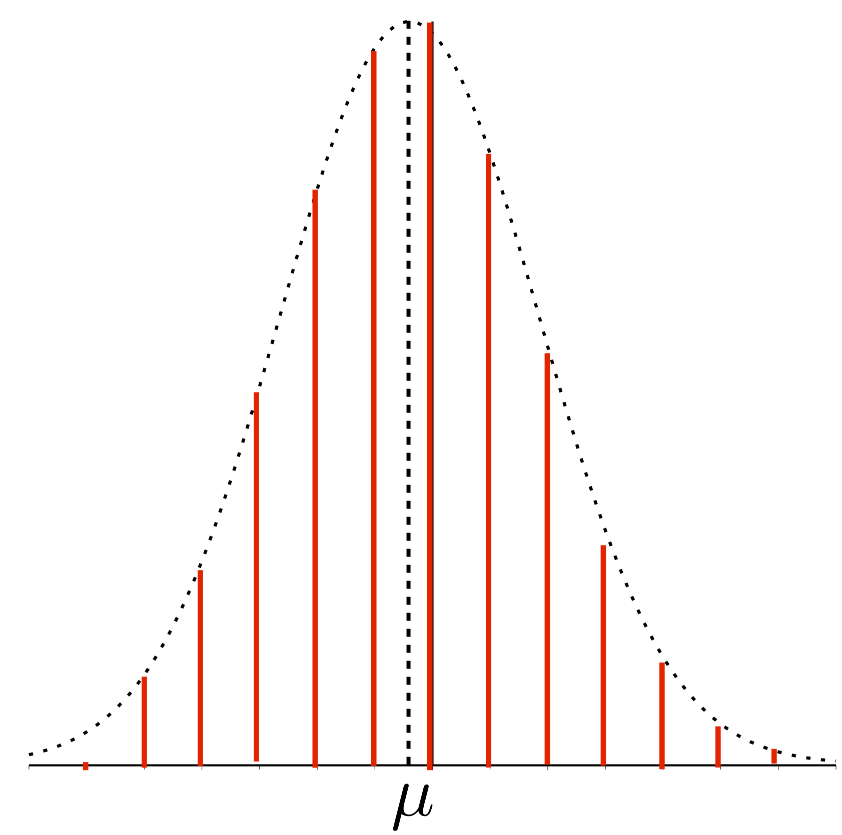
+ discrete Gaussian, centered at $\mu=-N\epsilon_\star \bmod \mathbb{Z}^g$

step $v[f]\propto\Big(\int_S\frac{f(x)\,x^i\,\mathrm{d}x}{\prod_\alpha|x-\alpha|^{1/2}}\Big)_{0\leq i\leq g-1}$

Corollary

$$\Big(\sum_{i=1}^N f(\lambda_i)-N\int_A f(\xi)\mathrm{d}\mu_{\mathrm{eq}}(\xi)\Big)$$

converges in law along subsequences



History in the 1-cut regime

$\beta = 2$

- If 1/N expansion exists, then $Z_N = N^{\gamma N + \gamma'} \exp\Big(\sum_{h \geq 0} N^{2-2h}\, F_{[h]}\Big)$
  and $F_{[h]}$ can be computed by the moment method
  Ambjørn, Chekhov, Kristjansen, Makeenko, 90s

- Rewriting of $F_{[h]}$ in terms of a universal topological recursion
  Eynard, '04

- Existence of 1/N expansion by
  - analysis of SD equations **Albeverio, Pastur, Shcherbina '01**
  - RH techniques **Ercolani, McLaughlin '02**
  - analysis of int. system **Bleher, Its, '05**

History in the 1-cut regime

$\beta > 0$

- if 1/N expansion exists, then

$$F_k = \sum_{h=0}^{\lfloor k/2 \rfloor + 1} \left(\frac{\beta}{2}\right)^{1-h} \left(1 - \frac{2}{\beta}\right)^{k+2-2h} F_{[h];k+2-2h}$$

and $F_{[h];m}$ computed by a β-topological recursion

Chekhov, Eynard '06

- Central limit theorem

Johansson '98

- Existence of 1/N expansion (analysis of SD eqn)

Borot, Guionnet '11

History in the (g + 1)-cuts regime

$\beta = 2$

- numerous observations of oscillatory behavior
  physicists, '90s
- asymptotics of $\langle \det(x - M) \rangle_{N \times N}$ up to o(1) (RH techniques)
  Deift, Kriecherbauer, McLaughlin, Venakides, Zhou '99

- heuristic derivation up to o(1)
  Bonnet, David, Eynard '00
- generalization to all orders
  Eynard '07

- observation of "no CLT"
  Pastur '06

$\beta > 0$

- Proof of "no CLT" and asymptotics of Z_N^A up to o(1)
  Shcherbina '12
- General proof
  Borot, Guionnet '13

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Orthogonal polynomials and random matrices

For $\beta = 2$

measure over the space of $N \times N$ hermitian matrices

$V(x) = V_0(x) + \sum_{j\geq 1} \frac{t_j}{j}\, x^j$

$\frac{1}{Z_{N,n}}\, \mathrm{d}M\, e^{-n\,\mathrm{Tr}\, V(M)}$

- $P_{N,n} = \mathbb{E}_{N\times N}\big[\det(x - M)\big]$

  is the Nth orthogonal polynomial for the weight $\mathrm{d}x\, e^{-nV(x)}$ on $\mathbb{R}$

- Let $h_{N,n}$ = norm of $P_{N,n}$

  $\widehat{P}_{N,n} = P_{N,n}/\sqrt{h_{N,n}}$ satisfies a 3-term recurrence relation

$$(x - \beta_{N,n})\widehat{P}_{N,n}(x) = \sqrt{h_{N,n}}\, \widehat{P}_{N+1,n}(x) + \sqrt{h_{N-1,n}}\, \widehat{P}_{N-1,n}(x)$$

Orthogonal polynomials and random matrices

For $\beta = 2$

measure over the space of $N \times N$ hermitian matrices

$$V(x) = V_0(x) + \sum_{j\geq 1} \frac{t_j}{j}\, x^j$$

$$\frac{1}{Z_{N,n}}\, \mathrm{d}M\, e^{-n\,\mathrm{Tr}\, V(M)}$$

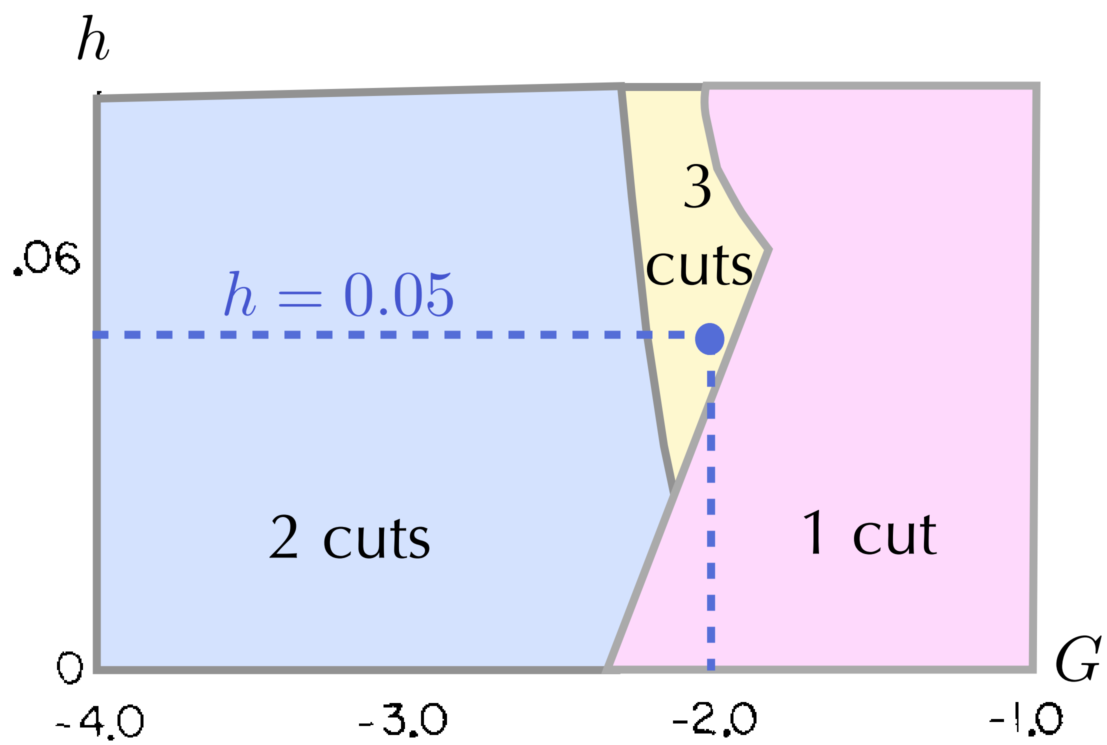
- The coefficients are solutions of a Toda chain :

$$\begin{cases} u_{N,n} = \ln h_{N,n} \\ v_{N,n} = -\beta_{N,n} \end{cases} \qquad \begin{cases} \partial_{t_1} u_{N,n} = v_{N,n} - v_{N-1,n} \\ \partial_{t_1} v_{N,n} = e^{u_{N+1,n}} - e^{u_{N,n}} \end{cases}$$

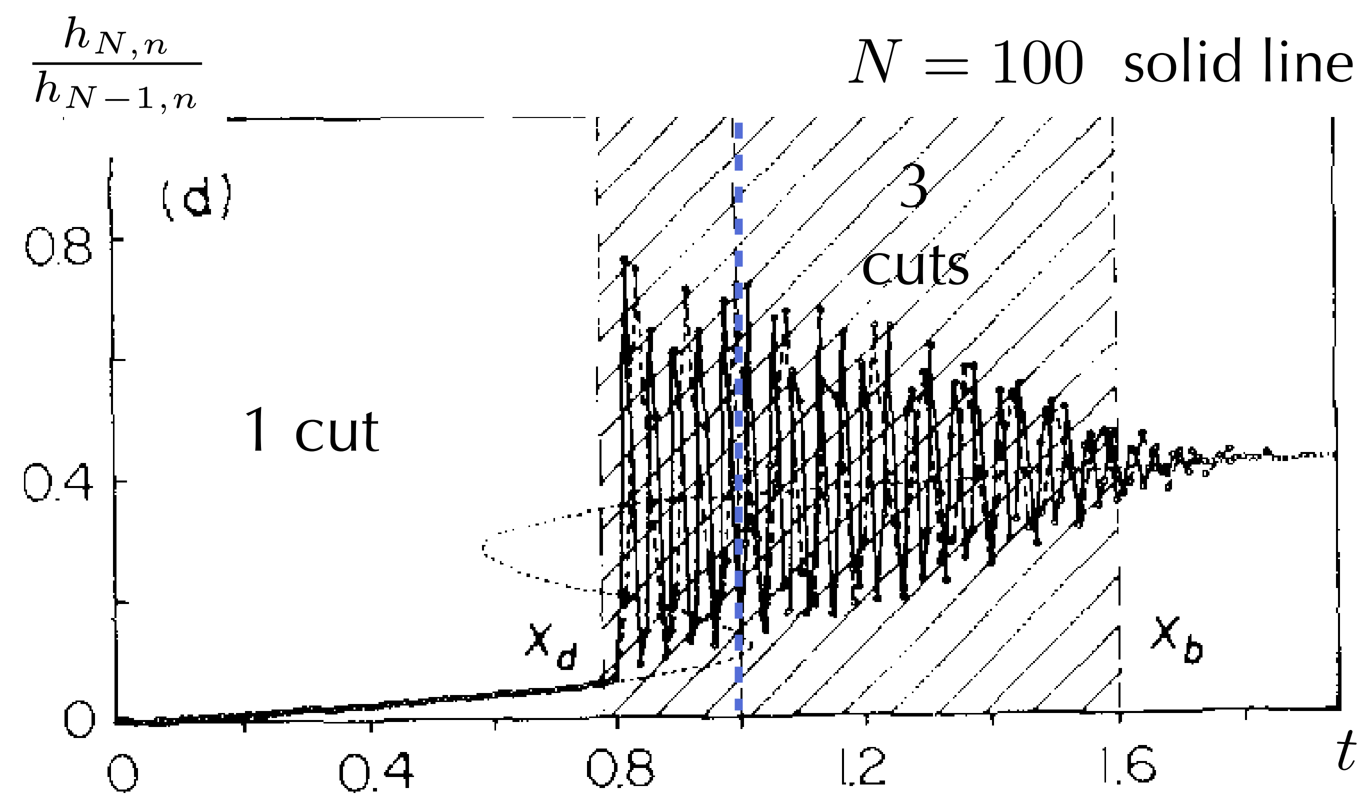
- ∂_{t_j} are the higher Toda flows
- initial condition prescribed by the string equations
- $Z_{N,n} = N! \prod_{j=0}^{N-1} h_{j,n}$ is the Tau function

The continuum limit of Toda

$N, n \to \infty \qquad N/n = t$ fixed



$V(x) = \frac{x^2}{2} + hG\,\frac{x^4}{4} + h^2\,\frac{x^6}{6}$



from Jurkiewicz '91 Phys. Lett. B, 261, 3

- if the model with V/t has (g + 1)-cuts and is off-critical

  main result & $h_{N,n} = \dfrac{1}{N+1}\,\dfrac{Z_{N+1,nN/(N+1)}}{Z_{N,n}}$

  $\Downarrow$

  all-order oscillatory asymptotics for $u_{N,n} = \ln h_{N,n}$

Asymptotics of orthogonal polynomials

$N, n \to \infty$
$N/n = t$ fixed

- main result + $P_{N,n}(x) = \dfrac{Z_{N,n}^{V-(1/N)\ln(x-\bullet)}}{Z_{N,n}^{V}}$

  $\Longrightarrow$ all-order asymptotics of $P_{N,n}(x)$ for x away from its zero locus

- $\beta = 1, 4$ are related to skew orthogonal polynomials/Pfaff latice

  $\langle P_{j,n} | P_{k,n} \rangle = (\delta_{j,k-1} - \delta_{j-1,k}) h_{j,n}$

$$\beta = 1 \quad \begin{cases} M = \textit{real symmetric} \\ \langle f | g \rangle_{\beta=1} = \int_{\mathbb{R}^2} \mathrm{d}x\mathrm{d}y\, e^{-n(V(x)+V(y))}\, \mathrm{sgn}(x-y)\, f(x) g(y) \\ N_{\beta=1} = 2N \end{cases}$$

$$\beta = 4 \quad \begin{cases} M = \textit{quaternionic self-dual} \\ \langle f | g \rangle_{\beta=4} = \int_{\mathbb{R}} \mathrm{d}x\, e^{-nV(x)} \big(f(x) g'(x) - f'(x) g(x) \big) \\ N_{\beta=4} = N \end{cases}$$

$$P_{2N,n}(x) = \mathbb{E}_{N_\beta \times N_\beta}\big[\det(x - M)\big]$$
$$P_{2N+1,n}(x) = \mathbb{E}_{N_\beta \times N_\beta}\big[(x + \mathrm{Tr}\, M)\det(x - M)\big]$$

$\Longrightarrow$ similar asymptotic results

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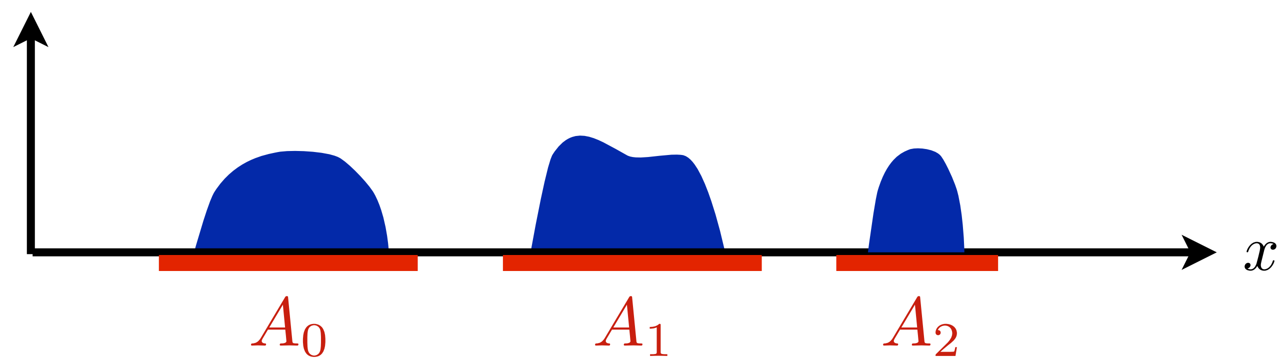
Conditioning on the filling fractions

- From local large deviations : up to $o(e^{-cN})$, we can choose



$$A = \bigcup_{h=0}^{g} A_h$$

- We will study $\mu_{(N_0,\ldots,N_g)}^{(A_0,\ldots,A_g)} = \mu_N^A$ conditioned to have $\begin{cases} N_0 \text{ first } \lambda\text{'s in } A_0 \\ N_1 \text{ next } \lambda\text{'s in } A_1 \\ \text{etc.} \end{cases}$

The partition function decomposes

$$Z_N^A = \sum_{N_0+\cdots N_g=N} \frac{N!}{\prod_{h=0}^{g} N_h!} Z_{(N_0,\ldots,N_g)}^{(A_0,\ldots,A_g)}$$

Correlators and partition function

- We will show a 1/N expansion for the m-point correlators :

$$W_m(x_1,\dots,x_m) = \mu_{\mathbf{N}}^{\mathbf{A}}\text{–cumulant}\Big(\sum_{i_1=1}^{N}\frac{1}{x_1-\lambda_{i_1}},\dots,\sum_{i_m=1}^{N}\frac{1}{x_m-\lambda_{i_m}}\Big)$$

$x_i \in \mathbb{C}\setminus A$

- If $(V_t)_t$ is a smooth family of potentials respecting our assumptions

$$\frac{Z_{\mathbf{N}}^{\mathbf{A};V_1}}{Z_{\mathbf{N}}^{\mathbf{A};V_0}} = \exp\Big[-N(\beta/2)\oint_A \frac{\mathrm{d}x}{2\mathrm{i}\pi}\partial_t V_t(x)\,W_1^{V_t}(x)\Big]$$ will have a large N expansion

- We need a reference V_0 where $Z_{\mathbf{N}}^{\mathbf{A};V_0}$ can be exactly computed

The Schwinger-Dyson equations

- Integration by parts $\Longrightarrow$ exact relations between $\mu_{\mathbf{N}}^{\mathbf{A}}$-cumulants

$$\int \prod_{1\leq i<j\leq N} |\lambda_i - \lambda_j|^\beta \prod_{i=1}^{N} e^{-N(\beta/2)V(\lambda_i)} \mathrm{d}\lambda_i$$

- E.g :

$$\mu_{\mathbf{N}}^{\mathbf{A}}\Big[\sum_{i=1}^{N} \frac{1}{(x-\lambda_i)^2} + \sum_{1\leq i<j\leq N} \frac{\beta}{(x-\lambda_i)(x-\lambda_j)} - \frac{N\beta}{2}\sum_{i=1}^{N}\frac{V'(\lambda_i)}{x-\lambda_i}\Big] + \sum_{a\in\partial A} \frac{\partial_a \ln Z_{\mathbf{N}}^{\mathbf{A}}}{x-a} = 0$$

which can be rewritten :

$$W_2(x,x) + \big(W_1(x)\big)^2 + (1-2/\beta)W_1'(x) - \oint_A \frac{\mathrm{d}\xi}{2\mathrm{i}\pi}\,\frac{V'(\xi)\,W_1(\xi)}{x-\xi} + \sum_{a\in\partial A}\frac{\partial_a \ln Z_{\mathbf{N}}^{\mathbf{A}}}{x-a} = 0$$

- For any $n \geq 1$, there is a quadratic relation between $W_{n+1}, W_n, \ldots, W_1$

A priori control on correlators

For the conditioned measure $\mu_{\mathbf{N}}^{\mathbf{A}}$

consider $N, (N_h)_h \to \infty$ with $\epsilon_h = N_h/N$ fixed, close enough to $\epsilon_h^{\star}$

- There is an equilibrium measure $\mu_{\mathrm{eq}}^{\epsilon}$ (depending smoothly on ϵ)

So : $N^{-1} W_1(x) \xrightarrow[N\infty]{} \int \frac{\mathrm{d}\mu_{\mathrm{eq}}(\xi)}{x - \xi}$

- From global large deviations :

$$\left| W_1(x) - N \int \tfrac{\mathrm{d}\mu_{\mathrm{eq}}^{\epsilon}(\xi)}{x-\xi} \right| \leq c_1[d(x, A)]\,(N \ln N)^{1/2}$$

$$\left| W_m(x_1, \ldots, x_m) \right| \leq \Big(\textstyle\prod_{i=1}^{m} c_m[d(x_i, A)] \Big) (N \ln N)^{m/2}$$

Rigidity of the Schwinger-Dyson equations

By recursive analysis of the Schwinger-Dyson equation :

$$\left| W_1(x) - N \int \frac{\mathrm{d}\mu_{\mathrm{eq}}^{\epsilon}(\xi)}{x-\xi} \right| \leq c_1[d(x,A)]\,(N \ln N)^{1/2}$$

$$\left| W_m(x_1,\ldots,x_m) \right| \leq \Big(\prod_{i=1}^{m} c_m[d(x_i,A)] \Big) (N \ln N)^{m/2}$$

$\Downarrow$ thanks to off-criticality

$$\Big(W_1(x) - N \int \frac{\mathrm{d}\mu_{\mathrm{eq}}^{\epsilon}(\xi)}{x-\xi} \Big) \longrightarrow W_1^{[0]}(x)$$

$$\left| W_m(x_1,\ldots,x_m) \right| \leq \Big(\prod_{i=1}^{m} c'_m[d(x_i,A)] \Big)\, N^{2-m}$$

concentration phenomenon

$$\Downarrow$$

$$W_m(x_1,\ldots,x_m) = \sum_{k \geq m-2} N^{-k}\, W_m^{[k]}(x_1,\ldots,x_m) + O(N^{-K})$$

for all K

(no uniformity)

Back to the partition function

$$\frac{Z_{\mathbf{N}}^{\mathbf{A};V_1}}{Z_{\mathbf{N}}^{\mathbf{A};V_0}} = \exp\Big[-(\beta/2) \sum_{k \geq -2} N^{-k} \oint_A \frac{\mathrm{d}x}{2\mathrm{i}\pi}\, \partial_t V_t(x)\, W_1^{V_t;[k+1]}(x) + O(N^{-(K-1)}) \Big]$$

To deduce an expansion for $Z_{\mathbf{N}}^{\mathbf{A};V}$, we need

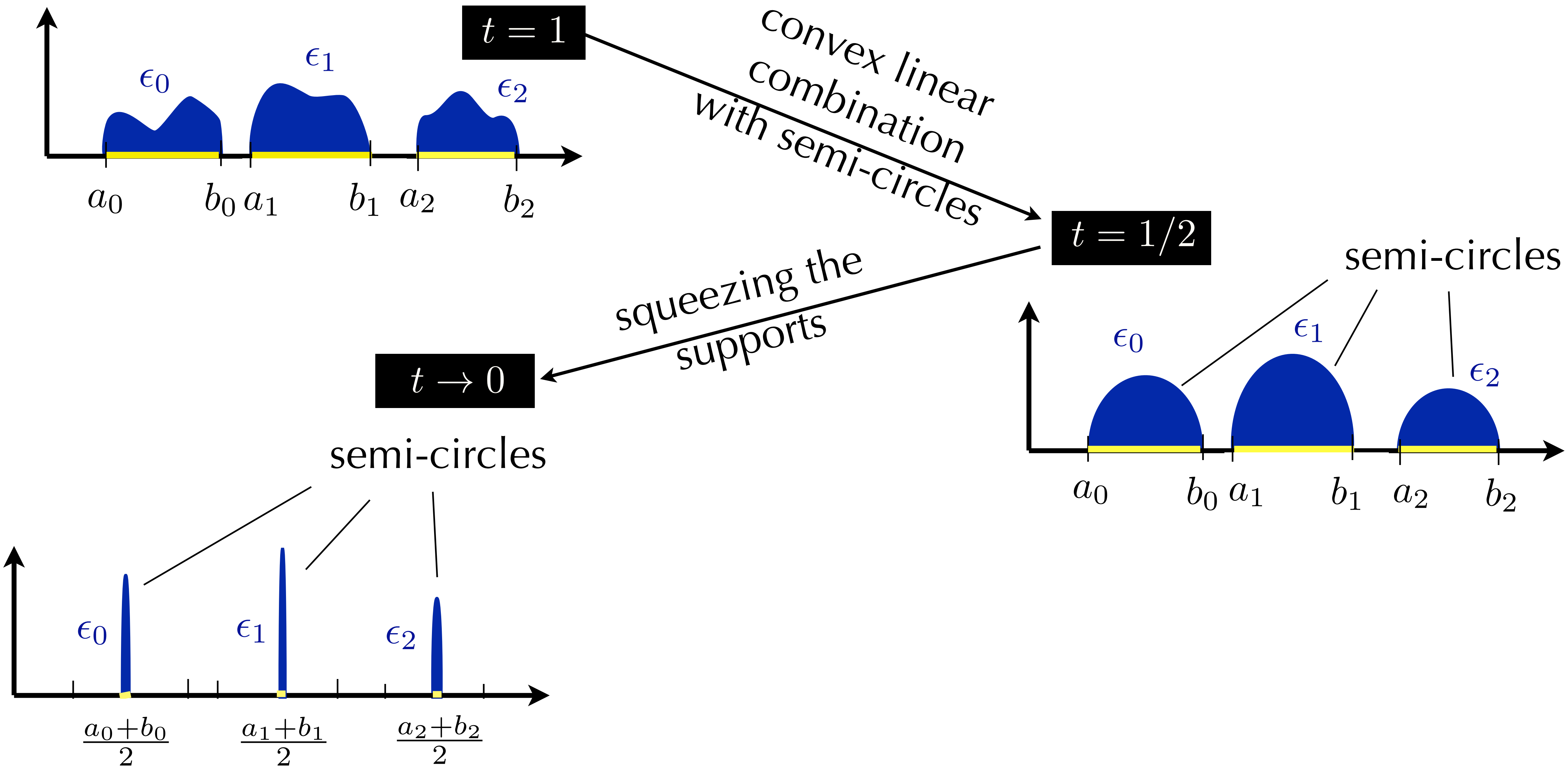
- V_0 such that $Z_{\mathbf{N}}^{\mathbf{A};V_0}$ is exactly known
- an interpolation $(V_t)_{t\in[0,1]}$ from $V_{t=1} = V$ staying uniformly (g + 1)-cuts and off-critical

Idea : interpolate in the space of equilibrium measures

$$(\mu_{\mathrm{eq}}^t)_{t\in[0,1]} \longleftrightarrow (V_t)_{t\in[0,1]}$$

$\int_A \ln|x-y|\mathrm{d}\mu_{\mathrm{eq}}^t(y) - V_t(x) = C_t$ with equality μ_{eq}^t-everywhere

An interpolation path ...



$$Z_N^A[V_t] \underset{t\to 0}{\sim} \prod_{0\leq h<h'\leq g} \left|\frac{a_h + b_h - a_{h'} - b_{h'}}{2}\right|^{N^2\epsilon_h\epsilon_{h'}\beta} \prod_{h=0}^{g} \left(\begin{array}{c}\text{Selberg } \beta\text{−Gaussian} \\ \text{integral over } \mathbb{R}^{N_h}\end{array}\right)$$

Sums and interferences - 1/3

We initially wanted to compute $$Z_N^A = \sum_{N_0+\cdots N_g=N} \frac{N!}{\prod_{h=0}^g N_h!} Z_{(N_0,\ldots,N_g)}^{(A_0,\ldots,A_g)}$$

- From global large deviations :

$$Z_N^A = \Big(\sum_{|\mathbf{N}-N\epsilon^\star|\leq \ln N} \frac{N!}{\prod_{h=0}^g N_h!} Z_{\mathbf{N}}^{\mathbf{A}} \Big)(1 + O(e^{-cN}))$$

- For $\mathbf{N} - N\epsilon^\star \in o(N)$, we just proved, with $\epsilon = (N_h/N)_{1\leq h\leq g}$

$$\frac{N!}{\prod_{h=0}^g N_h!} Z_{\mathbf{N}}^{\mathbf{A}} = N^{\gamma N + \gamma'} \exp\Big[\sum_{k\geq -2} N^{-k} F_k(\epsilon) + O(N^{-K}) \Big]$$

- Extra lemma : $F_k(\epsilon)$ are smooth functions of $\epsilon \approx \epsilon^\star$

$F'_{-2}(\epsilon^\star) = 0$ and $F''_{-2}(\epsilon^\star) < 0$

Sums and interferences - 2/3

We plug the asymptotic formula and use a Taylor expansion at $\epsilon \approx \epsilon^\star$

- E.g. up to o(1) :

$$Z_N^A = N^{\gamma N+\gamma'} e^{N^2 F_{-2}(\epsilon^\star)+NF_{-1}(\epsilon^\star)+F_0(\epsilon^\star)}$$
$$\times\Big(\sum_{|\mathbf{N}-N\epsilon^\star|\leq \ln N} e^{\frac{1}{2}F''_{-2}(\epsilon^\star)\cdot(\mathbf{N}-N\epsilon^\star)^{\otimes 2}+F'_{-1}(\epsilon^\star)\cdot(\mathbf{N}-N\epsilon^\star)}\Big)\big(1+O(e^{-c(\ln N)^3/N})\big)$$

It is the general term of a super-exponentially fast converging series :

$$Z_N^A = N^{\gamma N+\gamma'} e^{N^2 F_{-2}(\epsilon^\star)+NF_{-1}(\epsilon^\star)+F_0(\epsilon^\star)}$$
$$\times\Big(\sum_{\mathbf{N}\in\mathbb{Z}^g} e^{\frac{1}{2}F''_{-2}(\epsilon^\star)\cdot(\mathbf{N}-N\epsilon^\star)^{\otimes 2}+F'_{-1}(\epsilon^\star)\cdot(\mathbf{N}-N\epsilon^\star)}\Big)\big(1+O(e^{-c(\ln N)^2})\big)$$

- We recognize $\Theta_{-N\epsilon^\star}(F'_{-1}|F''_{-2})$

Sums and interferences - 3/3

- Including higher orders yields terms of the form

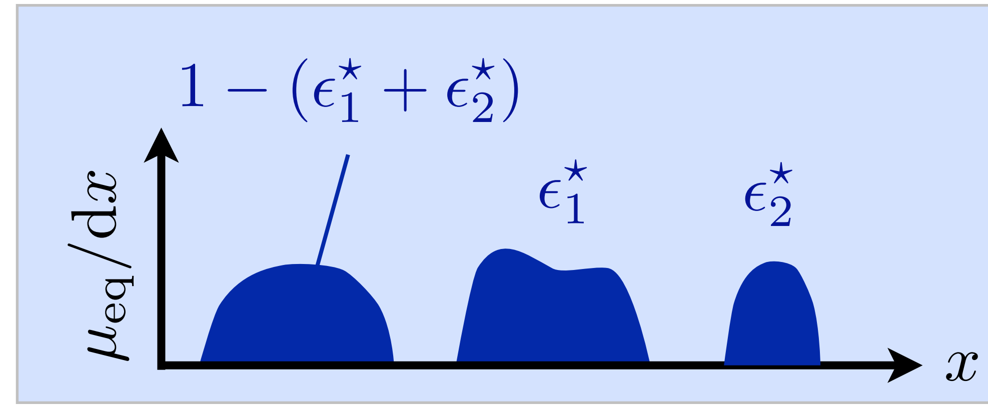
$$\sum_{\mathbf{N}\in\mathbb{Z}^g}\Big(\frac{1}{r!}\prod_{i=1}^{r}\frac{F_{k_i}^{(\ell_i)}(\epsilon^\star)\cdot(\mathbf{N}-N\epsilon^\star)^{\otimes\ell_i}}{\ell_i!}\Big)\,e^{\frac{1}{2}\mathbf{Q}\cdot(\mathbf{N}-N\epsilon^\star)^{\otimes 2}+\mathbf{w}\cdot(\mathbf{N}-N\epsilon^\star)}$$

We recognize $\Big(\frac{1}{r!}\prod_{i=1}^{r}\frac{F_{k_i}^{(\ell_i)}(\epsilon^\star)\cdot\nabla_{\mathbf{w}}^{\otimes\ell_i}}{\ell_i!}\Big)\Theta_{-N\epsilon^\star}(\mathbf{w}|\mathbf{Q})$

Here $\mathbf{Q}=F''_{-2}(\epsilon^\star)$ and $\mathbf{w}=F_{-1}(\epsilon^\star)$

- We justified step by step the heuristics of **Bonnet, David, Eynard '00, Eynard '07**

Summary : the (g + 1)-cuts regime



- Oscillatory asymptotic expansion

$$Z_N^A = N^{\gamma N+\gamma'}(\mathcal{D}_N\Theta_{-N\epsilon_\star})\big(F'_{-1}\big|F''_{-2}\big)\exp\Big[\sum_{k\geq -2} N^{-k}\,F_k + O(N^{-\infty})\Big]$$

where $\quad \mathcal{D}_N = \sum_{r\geq 0}\frac{1}{r!}\sum_{\substack{\ell_1,\ldots,\ell_r\geq 1\\ k_1,\ldots,k_r\geq -2\\ \sum_i(k_i+\ell_i)>0}} N^{-(\sum_i k_i+\ell_i)}\prod_{i=1}^{r}\frac{F_{k_i}^{(\ell_i)}\cdot\nabla_{\mathbf{w}}^{\otimes\ell_i}}{\ell_i!}$

acts as a differential operator on the Siegel theta function

$$\Theta_\mu(\mathbf{w}|\mathbf{Q}) = \sum_{\mathbf{m}\in\mathbb{Z}^g} e^{\mathbf{w}\cdot(\mathbf{m}+\mu)+\frac{1}{2}(\mathbf{m}+\mu)\cdot\mathbf{Q}\cdot(\mathbf{m}+\mu)}$$

- Moving characteristics $\qquad \mu = -N\epsilon_\star \bmod \mathbb{Z}^g$

  Quadratic form $\qquad \mathbf{Q} = F''_{-2} = 2\mathrm{i}\pi\,(\beta/2)\times(\text{period matrix}) < 0$

All order asymptotics for β-ensembles in the multi-cut regime

1. Beta-ensembles and random matrices

2. Applications to orthogonal polynomials

3. Ideas about the proof

4. Perspectives

Generalization ...

... to real-analytic k-body interactions

$$\mathrm{d}\mu_N^A = \prod_{1\leq i<j\leq N} |\lambda_i - \lambda_j|^\beta \exp\Big(\sum_{k=1}^{r} \frac{N^{2-k}}{k!} \sum_{i_1,\ldots,i_k=1}^{N} T_k(\lambda_{i_1},\ldots,\lambda_{i_k})\Big) \prod_{i=1}^{N} \mathrm{d}\lambda_i$$

- Equilibrium measure & local large deviations provided

  $\mathcal{E}[\mu] = \frac{\beta}{2} \iint \ln|x_1 - x_2| \mathrm{d}\mu(x_1)\mathrm{d}\mu(x_2) + \sum_{k=1}^{r} \frac{1}{k!} \int T_k(x_1,\ldots,x_k) \prod_{i=1}^{k} \mathrm{d}\mu(x_i)$

  has a unique minimum

- Global large deviations provided $\mathcal{E}''[\mu_{\mathrm{eq}}] < 0$
- Similar asymptotic results

} **in progress with Guionnet, Kozlowski**

- Coefs. of expansions are given by a "blobbed" topological recursion

  Borot '13

General ideas

- Nature of expansion depends on the topology of the spectrum

connected $\longrightarrow$ 1/N expansion

gaps $\longrightarrow$ ... + interference patterns

- Structure of expansion is influenced by singularities of the measure on the "moduli space" $\mathfrak{M} = A^N/\mathfrak{S}_N$

$\prod_{i<j} |\lambda_j - \lambda_j|^\beta$ = non-analyticity on $\partial\mathfrak{M}$

Open problems

- Singular V's and uniform asymptotics around critical points
  - asymptotics of (skew) orthogonal polynomials in the bulk
  - universality and computing tails of universal laws

- Complex-valued V
  - Berry-Esséen type estimates in CLT

- Same questions for $\lambda_i \in \mathbb{C}$

- Same questions for multi-matrix models
  - asymptotics of biorthogonal polynomials