Problem sheet 3, PCMI

Burt Totaro

July 2024, Park City, Utah

You could also look at problems on earlier problem sheets, if you like.

(1) Let $Q = Q_8$ be the quaternion group of order 8,

$$Q = \langle i, j | i^4 = 1,\ j^2 = i^2,\ jij^{-1} = i^{-1} \rangle.$$

Consider Q as a group scheme over $\mathbf{C}$.

(a) Show that Q has a faithful representation V of dimension 2 over $\mathbf{C}$. Show that Q acts freely on $V - 0$ (by inspection of your representation).

(b) Apply the localization sequence for equivariant Chow groups to $\{0\} \subset V$ to show that CH^*BQ is generated as a module over the polynomial ring $\mathbf{Z}[c_2V]$ by elements of degree ≤ 1.

(c) You may use the facts that $CH^1BG \cong \mathrm{Hom}(G, \mathbf{C}^*)$ for every finite group G and that

$$H^*(BQ, \mathbf{Z}) \cong \mathbf{Z}[a, b, c_2V]/(2a, 2b, 8c_2V, a^2, b^2, ab - 4c_2V).$$

Also using (b), compute the Chow ring of BQ.

(2) (a) Let G be an affine group scheme over a field k with a faithful representation V of dimension n. You may use that

$$CH^*(GL(n)/G) \cong CH^*BG/(c_1V, \ldots, c_nV).$$

Deduce that CH^*BG is generated as a module over $\mathbf{Z}[c_1V, \ldots, c_nV]$ by elements of degree at most $n^2 - \dim(G)$.

(b) Show that $GL(n)/O(n)$ over $\mathbf{C}$ is isomorphic to a Zariski open subset of affine space. (Note that $O(n)$ here denotes the complex orthogonal group, the subgroup of $GL(n)_{\mathbf{C}}$ that preserves the symmetric bilinear form on $\mathbf{C}^n$ given by $\langle e_i, e_j \rangle = \delta_{ij}$. The maximal compact subgroup of $O(n)_{\mathbf{C}}$ is the compact real orthogonal group $O(n)$, so they're homotopy equivalent, but they're not the same group.) (Hint: consider the action of $GL(n)$ on the vector space of symmetric bilinear forms on $\mathbf{C}^n$.)

(c) Using (a) and (b), show that the Chow ring of $BO(n)$ is generated by the Chern classes of the standard n-dimensional representation V of $O(n)$.

(d) More precisely, show that

$$CH^*BO(n) \cong \mathbf{Z}[c_1V, \ldots, c_nV]/(2c_i = 0 \text{ for } i \text{ odd}).$$

(Use whatever methods you can. E.g., what do you get from the fact that V is self-dual as a representation of $O(n)$? It may also be useful to remember what $H^*(BO(n), \mathbf{Q})$ and $H^*(BO(n), \mathbf{Z}/2)$ are, from Milnor-Stasheff's *Characteristic Classes*.)