Problem sheet 2, PCMI

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You could also look at problems on problem sheet 1, if you like.

(1) Let G be the multiplicative group G_m over a field k. Let G act on the affine plane A^2 over k by

$$t(x, y) = (tx, t^{-1}y).$$

Compute the G-equivariant Chow ring $CH^*_G(A^2 - 0)$.

(Side question: What is the geometric quotient $(A^2 - 0)/G$? This is not a separated scheme, so it's outside the usual setting where I defined Chow groups.)

(2) Let $G = \mathbf{Z}/3 = \langle g : g^3 = 1\rangle$, as an algebraic group over $\mathbf{C}$. Let G act on A^2 over $\mathbf{C}$ by

$$\sigma(x, y) = (\zeta x, \zeta^{-1}y),$$

where $\zeta := \zeta_3 = e^{2\pi i/3}$, a primitive cube root of unity. Show how to identify $X := A^2/G$ with a closed subvariety of A^N for some N (and find its equations explicitly). Find the singular locus of X from your equations. Compute the Chow groups of A^2/G. (Maybe first compute the Chow groups of $(A^2 - 0)/G$ using equivariant Chow groups; this is simpler in that G acts freely on $A^2 - 0$.)

(3) Let X be a smooth projective curve of genus $g \geq 1$ over $\mathbf{C}$. Show that the Chow-Künneth property fails for X; more precisely, show that $CH^*X \otimes_{\mathbf{Z}} CH^*X \to CH^*(X \times X)$ is not surjective, even after tensoring with $\mathbf{Q}$. (Hint: map "everything" into integral cohomology. Show that the image of $CH^*X \otimes_{\mathbf{Z}} CH^*X$ in $H^*(X \times X, \mathbf{Z})$ is fairly small. Show that the class of the diagonal Δ_X in $X \times X$ is not in the image of the product map, by computing intersection numbers.)