Making Contact with the Sphere

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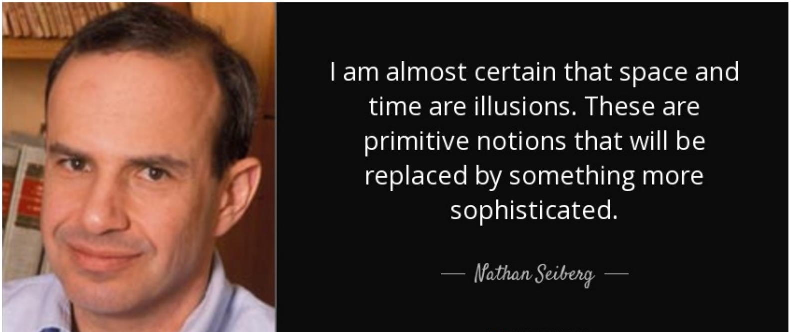


I am sure this is a correct, deep thought. My two cents:

When I am almost certain that I have some nice result or idea and I send it to Nati for scrutiny, I often find out that it was too primitive and Nati quickly replaces it by something more sophisticated and exciting.

When I arrived at the IAS Nati asked me some questions. I even remember the first question, on my first day:

- Which country is bigger, the USA or Israel?

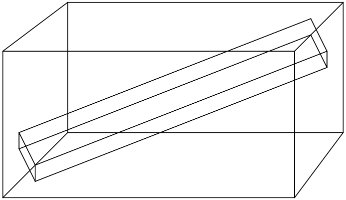
The questions were gradually becoming harder and I am still working on some of them....

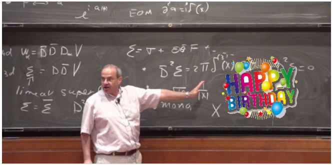
However, as I was preparing for the trip it occurred to me that I should probably muster enough courage and ask my own question.

Guiding Principle: *A distinguishing quality of Israelis is that they don't like to be cheated or fooled.*

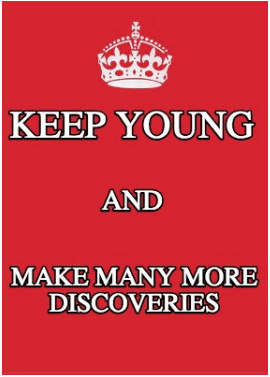
When I thought about using a courier service to send Nati a gift, it turned out the cost of sending a rectangular box is determined by the sum of its dimensions; that is, length plus width plus height. I immediately felt that I might be cheated. So I thought about packing a box into a cheaper box!!

Question to Nati (and everybody else): Did I find a way to pack a box into a cheaper one?





Happy Birthday and Many Many Happy Returns!



The purpose of this talk is to re-visit some issues in four-dimensional $\mathcal{N} = 2$ theories. Our results have implications for geometry, integrability, bootstrap, and resurgence theory. They also carry over to two-dimensional $(2, 2)$ theories more or less directly.

In conformal field theories, we have the so-called primary operators

$$\mathcal{U}_i(0)\ .$$

They transform nicely under conformal transformations. E.g. under $x \to \lambda x$, $ds^2 \to \lambda^2 ds^2$ and

$$\mathcal{U}_i(0) \to \lambda^{-\Delta_i}\mathcal{U}_i(0)\ .$$

More generally, since the physics in conformal field theories is independent of the scale but only depends on angles, we can study conformal field theories on spaces with the metric

$$ds^2 = \Omega^2(x) ds^2_{\mathbb{R}^d}$$

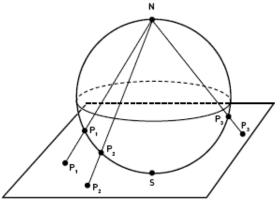
As an example we can choose $\Omega^2(x) = \frac{4}{(1+x^2)^2}$ and then our space is $\mathbb{S}^d$ with radius 1. Therefore, the physics of conformal field theories on $\mathbb{S}^d$ should be the same as the physics in flat space.

A consequence is that primary operators can be mapped from $\mathbb{R}^d$ to $\mathbb{S}^d$ via the stereographic projection

$$\mathcal{U}_i \to \Omega^{-\Delta_i}\mathcal{U}_i$$

And since in flat space $\langle \mathcal{U} \rangle_{\mathbb{R}^d} = 0$ we are led to conclude that

$$\langle \mathcal{U} \rangle_{\mathbb{S}^d} = 0$$



However on the sphere there is a dimensionful parameter R and so in principle we could get

$$\langle \mathcal{U} \rangle_{\mathbb{S}^d} = R^{-\Delta_i}$$

Let us interpret this as mixing of $\mathcal{U}_i$ with the unit operator.

How badly can we violate our expectation that the physics of conformal field theories does not depend on $\Omega^2(x)$?

General principles allow only violations that can be traced back to contact terms in $\mathbb{R}^d$. Therefore, the most general operator mixing on $\mathbb{S}^d$ takes the form

$$\mathcal{U}_\Delta \to \mathcal{U}_\Delta + R^{-2}\mathcal{U}_{\Delta-2} + \cdots + R^{-4}\mathcal{U}_{\Delta-4} + \ldots$$

Mixing with the unit operator can thus occur only if Δ is an even integer.

Operator mixing is associated to contact terms like

$$\mathcal{U}_\Delta T_{\mu\nu} \sim \partial^{2n}\delta^{(d)}(x)\mathcal{U}_{\Delta-2n}$$

etc. Upon coupling the theory to curved space, these contact terms complicate the transformation laws of $\mathcal{U}$ and hence induce the above mixing.

In most cases these contact terms are tunable and we can choose them to be whatever we like. A particularly convenient choice is to demand

$$\langle \mathcal{U}_\Delta(South)\mathcal{U}_{\Delta'}(North)\rangle_{\mathbb{S}^d} \sim \delta_{\Delta,\Delta'} \ .$$

(And, in particular, in this scheme the one-point functions vanish for operators with nonzero dimension $\langle \mathcal{U}\rangle_{\mathbb{S}^d} = 0$.)

There are some interesting situations where a scheme where

$$\langle \mathcal{U}_\Delta(South)\mathcal{U}_{\Delta'}(North)\rangle_{\mathbb{S}^d} \sim \delta_{\Delta,\Delta'} \quad (*) \ .$$

Does not exist. This can be viewed as a generalized trace anomaly. There is no complete classification of those, but we will see some examples soon.

Another important general point is that sometimes we may not like to allow some contact term

$$\mathcal{U}_\Delta T_{\mu\nu} \sim \partial^{2n}\delta^{(d)}(x)\mathcal{U}_{\Delta-2n}$$

because it may break a symmetry, e.g. SUSY. Then, there won't generally exist a scheme where $(*)$ holds.

Now I would like to remind of another important concept. Some conformal field theories have exactly marginal operators $\mathcal{U}_A$. These are operators that have dimension $\Delta(\mathcal{U}_A) = d$ and furthermore, if we add them to the action

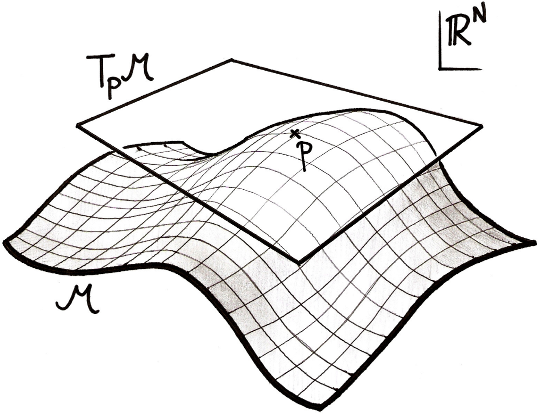
$$\delta S = \int d^d x \sum_A \lambda^A \mathcal{U}_A(x)$$

the theory remains conformal ($\beta^A = 0$). Then we have a manifold $\mathcal{M}$ of conformal field theories and $A = 1, .., dim\mathcal{M}$.

This situation is quite familiar, e.g. it has been observed long time ago in the Ashkin-Teller model.

Zamolodchikov introduced a metric on the space of theories $\mathcal{M}$:

$$\langle \mathcal{U}_A(0)\mathcal{U}_B(\infty)\rangle_{\mathbb{R}^d} = g_{AB}$$



Exactly marginal operators are commonplace in supersymmetric theories. They appear, e.g., in the theories studied by Seiberg and Witten: $SU(N)$ gauge group with $N_f = 2N_c$ fundamental hypermultiplets.

The metric on the space of these theories as well as the metric on various naturally associated holomorphic bundles can be determined using the tools we introduced above.

We will also see that the metrics on these spaces are subject to various recursion relations that are rooted in integrable models.

In $\mathcal{N} = 2$ superconformal theories it is useful to study the chiral primary operators

$$\bar{Q}^{1,2}\mathcal{O} = 0 \ .$$

These operators have a well-defined dimension and $U(1)_R$ charge

$$\Delta = \frac{1}{2}R \ .$$

We refer to the set of all such operators as the chiral ring.

Here we will try to study the "minimal" non-holomorphic observables in the theory, i.e. the extremal correlators

$$\langle O_{I_1}(x_1)O_{I_2}(x_2)...O_{I_n}(x_n)\bar{O}_J(w)\rangle$$

The correlation function may be non-vanishing if

$$\sum_{k=1}^{n}\Delta_{I_k}=\Delta_J\ .$$

The matrix of two-point functions of chiral operators with $\Delta = 2$ is the Zamolodchikov metric.

The higher correlation functions of chiral operators can be said to furnish a holomorphic vector bundle and their two-point functions define a Hermitian metric on this space.

If we know all the 2-point functions and chiral ring relations, then all the extremal correlators are determined. By no loss of generality we can study

$$\langle O_{I_1}(x_1) O_{I_2}(x_2) ... O_{I_n}(x_n) \bar{O}_J(\infty) \rangle$$

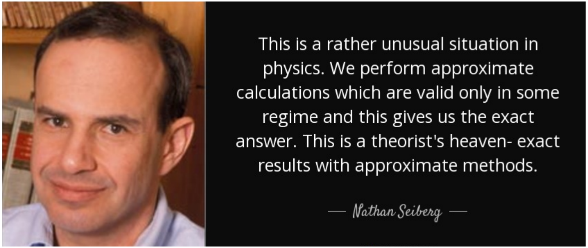
with $O(\infty) = \lim_{x\to\infty} x^{2\Delta_O} O(x)$ as usual.

Imagine doing the OPE for the string $O_{I_1} \dots O_{I_n}$. To get a nontrivial result we need an operator of dimension Δ_J on the right hand side. By the BPS bound it has to be the chiral ring element that we get by picking the leading term. Hence, the correlation function is independent of the x_i and thus fixed by the chiral ring relations and the Hermitian metric.

One can keep in mind 3 examples

- $\mathcal{N} = 4$ theory. Argued by [Lee-Minwalla-Rangamani-Seiberg] that the extremal correlation functions are exact at tree-level (for any N).

  This allowed interesting checks of the AdS/CFT duality.

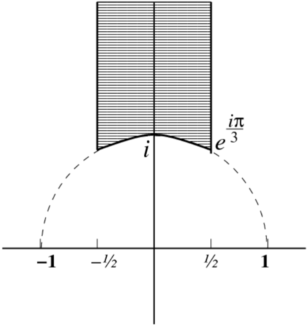


- Seiberg-Witten theory, i.e. $SU(2)$ with four fundamental hypermultiplets. The chiral ring consists of the operators

$$\mathcal{O}_n = \left(Tr(\varphi^2)\right)^n \ .$$

We have an exactly marginal coupling $\tau = \frac{\theta}{2\pi} + \frac{4\pi i}{g_{YM}^2}$. The chiral ring relations are

$$\mathcal{O}_n \mathcal{O}_m \sim \mathcal{O}_{n+m}$$



The correlator $\langle Tr(\varphi^2)\overline{Tr(\varphi^2)}\rangle = G_2(\tau,\bar\tau)$ corresponds to the Zamolodchikov metric on this space.
More generally, we define

$$\langle \mathcal{O}_n \overline{\mathcal{O}}_n \rangle = G_{2n}(\tau,\bar\tau)\ .$$

This can be thought of as the metric on some natural holomorphic vector bundle.

Unlike in $\mathcal{N}=4$ or the Ashkin-Teller model, the metric is not just the Poincaré metric on the upper half plane! For example, around weak coupling, we find a nontrivial perturbative expansion

$$G_2 = \frac{6}{(Im\tau)^2} - \frac{135\zeta(3)}{2\pi^2}\frac{1}{(Im\tau)^4} + \frac{1575\zeta(5)}{4\pi^3}\frac{1}{(Im\tau)^5} + \cdots \ .$$

Based on an explicit computation by [Baggio-Niarchos-Papadodimas].

In $SU(N)$ with $N_f = 2N$ fundamental hypermultiplets the situation is quite similar but with two important differences

- The chiral ring is freely generated by $N-1$ operators,

$$\mathcal{O}_{\vec{n}} = \prod_{k=2}^{N} \left(Tr(\varphi^k) \right)^{n_k}$$

- The duality group is slightly different and so is the fundamental domain.

The metric is measured by

$$\langle Tr(\varphi^2) \overline{Tr(\varphi^2)} \rangle$$

We start with the claim that:

$$\langle Tr(\varphi^2)\rangle_{\mathbb{S}^4} = \partial_\tau K(\tau,\bar\tau)$$

where $ds^2 = \partial_\tau\partial_{\bar\tau}K(\tau,\bar\tau)d\tau d\bar\tau$ is the Zamolodchikov metric.

We see that the mixing with the unit operator is **physical** and it is related to a certain new anomaly. The anomaly is essentially the $\mathcal{N}=2$ version of the "Paneitz anomaly"

$$\delta_\Sigma \log Z = \int d^4x d^8\theta (\Sigma + \bar\Sigma) K(\tau,\bar\tau) \ .$$

The Paneitz operator is a conformally invariant $\hat\Box^2$ in four dimensions.

On the other hand, a little bit of work with SUSY Ward identities shows that $\langle Tr(\varphi^2)\rangle_{\mathbb{S}^4} = \partial_\tau \log Z_{\mathbb{S}^4}$.
Naively, a τ derivative lowers **integrated top** components rather than **localized bottom** components. The top component vanishes essentially when we can write it as $Q^2(\textit{something})$, which can be done away from one point. So we only have to perform a careful analysis around that point.

The conclusion is that

$$Z_{\mathbb{S}^4} = e^{K(\tau,\bar{\tau})} \ .$$

$$Z_{\mathbb{S}^4} = e^{K(\tau,\bar{\tau})}$$

leads to consequences for geometry:

- K must be well defined!! $\longrightarrow$ no cycles in the space of theories, $[\omega] = 0$. (The Hodge bundle has trivial Chern Class.) Must be non-compact.
- Same holds in $d = 2$ where the metric on the moduli space of Calabi-Yau sigma models is the standard Weil-Petersson metric. We see that the Hodge bundle of the moduli space must have vanishing Chern Class.
- So in $d = 2$ we are talking about the standard space of CY manifolds while in $d = 4$ we make contact with the "quantum" moduli space of Riemann surfaces with punctures [Gaiotto], which are parameterized by $\tau, \bar{\tau}$.
- These new mathematical results about moduli spaces seem to be correct!

Raises some open questions

- We see that $\mathcal{M}$ is non-compact. But in simple examples we observe that it has finite volume. Is this always the case?
- (Related) Are the only non-compact directions associated to weak gauging?
- Is the curvature always non-positive?

Let us continue studying $SU(2)$ SQCD or $\mathcal{N}=4$ $SU(2)$ theory. We can generate *normalized* correlation functions on $\mathbb{S}^4$ by

$$\left\langle \left(Tr(\varphi^2)\right)^m (N) \left(\overline{Tr(\varphi^2)}\right)^n (S) \right\rangle_{\mathbb{S}^4} = \frac{1}{Z_{\mathbb{S}^4}} \frac{\partial^m}{\partial \tau^m} \frac{\partial^n}{\partial \bar{\tau}^n} Z_{\mathbb{S}^4}$$

Because of operator mixing these are not diagonal in m, n. Due to our observations above concerning the case $m=1, n=0$ we can write

$$\left\langle \left(Tr(\varphi^2)\right)^m (N) \left(\overline{Tr(\varphi^2)}\right)^n (S) \right\rangle_{\mathbb{S}^4} = e^{-K} \frac{\partial^m}{\partial \tau^m} \frac{\partial^n}{\partial \bar{\tau}^n} e^K$$

We are ultimately interested in these two point functions in flat space. For this we need to systematically remove the operator mixing that the sphere induces. We thus need to use the Gram-Schmidt procedure.

Let us do the simplest example: the two-by-two case involving the unit operator and a $\Delta = 2$ chiral primary:

$$\begin{pmatrix} \langle 1 \rangle_{\mathbb{S}^4} & \langle Tr(\varphi^2)(N) \rangle_{\mathbb{S}^4} \\ \langle \overline{Tr(\varphi^2)}(S) \rangle_{\mathbb{S}^4} & \langle Tr(\varphi^2)(N)\overline{Tr(\varphi^2)}(S) \rangle_{\mathbb{S}^4} \end{pmatrix} = \frac{1}{Z_{\mathbb{S}^4}} \begin{pmatrix} Z_{\mathbb{S}^4} & \partial_\tau Z_{\mathbb{S}^4} \\ \partial_{\bar{\tau}} Z_{\mathbb{S}^4} & \partial_\tau \partial_{\bar{\tau}} Z_{\mathbb{S}^4} \end{pmatrix}$$

The Gram-Schmidt procedure amounts to taking the determinant

$$\langle Tr(\varphi^2)(0)\overline{Tr(\varphi^2)}(\infty) \rangle_{\mathbb{R}^4} = \det\left[\frac{1}{Z_{\mathbb{S}^4}} \begin{pmatrix} Z_{\mathbb{S}^4} & \partial_\tau Z_{\mathbb{S}^4} \\ \partial_{\bar{\tau}} Z_{\mathbb{S}^4} & \partial_\tau \partial_{\bar{\tau}} Z_{\mathbb{S}^4} \end{pmatrix} \right]$$

$$= \partial_\tau \partial_{\bar{\tau}} \log Z_{\mathbb{S}^4} = g_{\tau\bar{\tau}} \ ,$$

as needed.

Using Pestun's computation of $Z_{\mathbb{S}^4}$ and the facts we explained we can thus now completely determine all the extremal correlators in essentially all $\mathcal{N}=2$ theories.

For example here is the one-instanton prediction for the Zamolodchikov metric in Seiberg-Witten theory:

$$\cos(\theta) e^{-\frac{8\pi^2}{g^2}} \left(\frac{6}{(Im\tau)^2} + \frac{3}{\pi}\frac{1}{(Im\tau)^3} - \frac{135\zeta(3)}{2\pi^2}\frac{1}{(Im\tau)^4} + \cdots \right) .$$

All the other extremal correlators in flat space are given by similar ratios of determinants:

$$G_{2m} = \left\langle \left(Tr(\varphi^2)\right)^m (0) \left(\overline{Tr(\varphi^2)}\right)^m (\infty) \right\rangle_{\mathbb{R}^4}$$

$$= \frac{1}{Z_{\mathbb{S}^4}} \frac{\det_{(k,l)=0,..,m} \partial_\tau^k \partial_{\bar{\tau}}^l Z_{\mathbb{S}^4}}{\det_{(k,l)=0,..,m-1} \partial_\tau^k \partial_{\bar{\tau}}^l Z_{\mathbb{S}^4}}$$

Such expressions of the chiral ring data in terms of ratios of determinants of derivatives of the sphere partition function generalize to any $\mathcal{N} = 2$ theory.

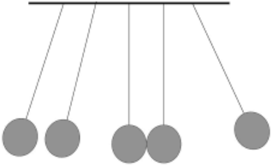
So we have arrived at a prescription that allows to compute essentially any extremal correlator in $\mathcal{N} = 2$ theories.

It has been appreciated for a long time that ratios of such determinants appear naturally as solutions of integrable models. Indeed, in simple examples we can now identify the integrable model which governs the chiral ring. In more complicated theories the identification is still an open problem.

- In $SU(2)$ we get an open Toda chain (in agreement with the non-perturbative tt^* equations, as developed in $d = 4$ by [Papadodimas] and solved by [Baggio-Niarchos-Papadodimas]), i.e.

$$\partial_\tau \partial_{\bar{\tau}} q_n = e^{q_{n+1} - q_n} - e^{q_n - q_{n-1}} \ ,$$

$q_n = \log(G_{2n} Z_{\mathbb{S}^4})$.



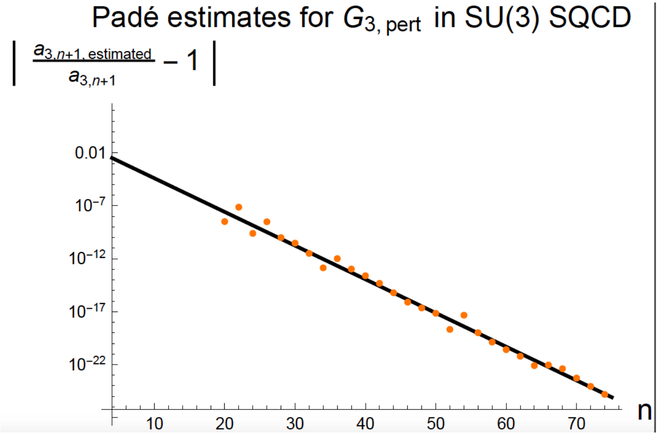
The difference between $SU(2)$ $\mathcal{N} = 2$ and $SU(2)$ $\mathcal{N} = 4$ is only in the boundary conditions at the open end of the Toda chain.

- More generally every $\mathcal{N}=2$ theory gives some integrable system! (This is proven.)
- The integrable system of $\mathcal{N}=4$ consists of infinitely many decoupled Toda chains.
- SQCD with $SU(N>2)$ gives coupled Toda chains of some sort. We can compute in principle everything to arbitrary order, e.g.

$$\langle Tr(\varphi^3)(0)\overline{Tr(\varphi^3)}(\infty)\rangle_{\mathbb{R}^4}$$
$$=\left(\frac{g^2}{4\pi}\right)^3\left(40-\frac{135\zeta(3)}{2\pi^4}g^4+\frac{6275\zeta(5)}{48\pi^6}g^6+\cdots\right)$$

[To understand better instanton contributions to extremal correlators in theories with irrelevant generators in the chiral ring, we need to slightly extend what is currently known about Omega background partition functions.]

We can check some general ideas about Quantum Field Theory, such as the convergence of the Padé approximation and Borel resummation.



Summary of the $d = 4$ $\mathcal{N} = 2$ part

- The chiral rings of $\mathcal{N} = 2$ theories furnish integrable systems. In the simplest cases we get Toda chains. But in general it's more complicated. For example, in $SU(N > 2)$ SQCD one finds a Hitchin-type system.
- Can test ideas about Borel summability, convergence of Padé approximations etc. Some recent rigorous proof of Borel summability by [Honda]
- Applications for AdS/CFT. Also can now solve many large N theories, e.g. SQCD in the Veneziano limit. Interesting progress by [Rodriguez-Gomez and Russo]
- Many open questions remain. In particular, it would be nice to understand the expectation values of higher Casimirs in the Omega background.

Aspects of two-dimensional $(2, 2)$ Theories.

Let us now make some comments about $(2,2)$ theories in $d=2$. An interesting family of SCFTs are Calabi-Yau sigma models. The space of exactly marginal parameters is the space of complex and Kähler structures. These correspond to chiral and twisted chiral exactly marginal operators.

- $\mathcal{M} = \mathcal{M}_{chiral} \times \mathcal{M}_{twisted-chiral}$.

[...Dixon-Kaplunovsky-Louis...]

One can also prove rather straightforwardly this factorization theorem using Seiberg's method of promoting couplings to superfields.

There are two inequivalent ways of compactifying such theories on S^2. The difference is very subtle: it affects only redundant operators. However, there is some generalization of a trace anomaly and one finds that the answer depends on these redundant terms (other derivations by [Gomis-Lee and Jockers et al.])

$$Z^A_{S^2} = e^{-K_{twisted-chiral}} , \qquad Z^B_{S^2} = e^{-K_{chiral}} .$$

The chiral ring is not freely generated. Quite the contrary: all the elements must be nilpotent for otherwise we would violate Coleman's theorem.

Operator mixing on S^2 still takes place and thus we still have our Gram-Shcmidt procedure and thus the connection to integrability and tt* equations remains.

We can solve for the extremal correlators in any $(2,2)$ SCFT as in $d = 4$. This has not been explored yet in great detail but in simple examples it was checked that this is in agreement with methods from the 90s.

A central puzzle is that while the Ward identities of $(2,2)$ supersymmetry imply that the moduli space of SCFTs is of the type $\mathcal{M} = \mathcal{M}_{chiral} \times \mathcal{M}_{twisted-chiral}$, this simple result fails in examples like $K3, \mathbb{T}^{2n}$ (for $n > 1$) etc.

For example (locally)

$$\mathcal{M}_{K3} = \frac{O(4,20)}{O(4) \times O(20)} \ ,$$

not only it is not Kähler, there is not even an invariant complex structure.

On the other hand, K3 is certainly a (2,2) sigma model and so one runs into contradictions with the above-stated results.

The idea is that there is a new anomaly that only exists if the (2,2) is a subalgebra of a larger superconformal algebra with a bigger R-symmetry.

One can prove a theorem that this is a *necessary and sufficient* condition for this "anomaly."

One can view it is an anomaly in spurion analysis – couplings that cannot be promoted to superfields. This is a technique that Nati used extremely successfully in the past. Now we see that there are some insteresting situations when it can fail!