Emergent Supersymmetry from a Lattice of Interacting Majorana Modes

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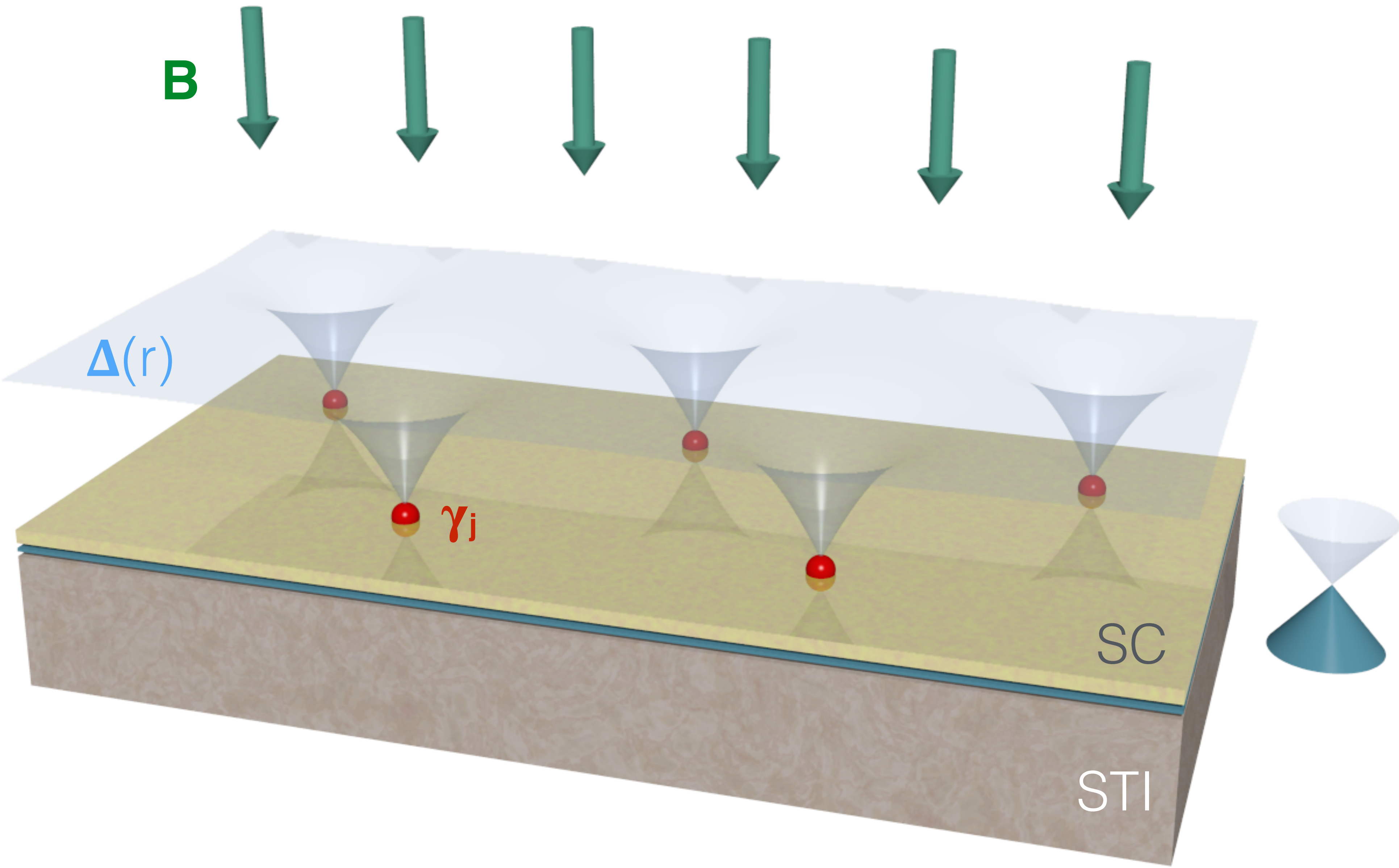
I) Introduction and Motivation

$$H = \sum_j [it\gamma_j \gamma_{j+1} + g\gamma_j \gamma_{j+1} \gamma_{j+2} \gamma_{j+3}],$$

$$\gamma_j^+ = \gamma_j, \quad \{\gamma_j, \gamma_l\} = 2\delta_{j,l}$$

- No conserved particle number but important discrete symmetries
- Not Bethe ansatz integrable?
- Can be studied by field theory and DMRG

- Experimental motivation, for D=1 or 2, is provided, for example, by a superconducting film on top of a strong topological insulator in a magnetic field which produces a vortex lattice
- A Majorana mode is localized at the core of each vortex
- In general tunneling between vortices is possible and Coulomb interactions are present
- An extra symmetry is present when t=0: $\gamma_j \rightarrow (-1)^j \gamma_j$
- This symmetry is present when the chemical potential of the topological insulator is tuned to the Dirac point of the surface states
- Could allow study of the strong g/t regime


B
Δ(r)
γj
SC
STI

Possibly either sign of g might occur experimentally since we are in a superconductor – we studied both signs.

II) Weak coupling phase diagram from field theory/RG methods

Non-interacting model is trivial to diagonalize: just Fourier transform:

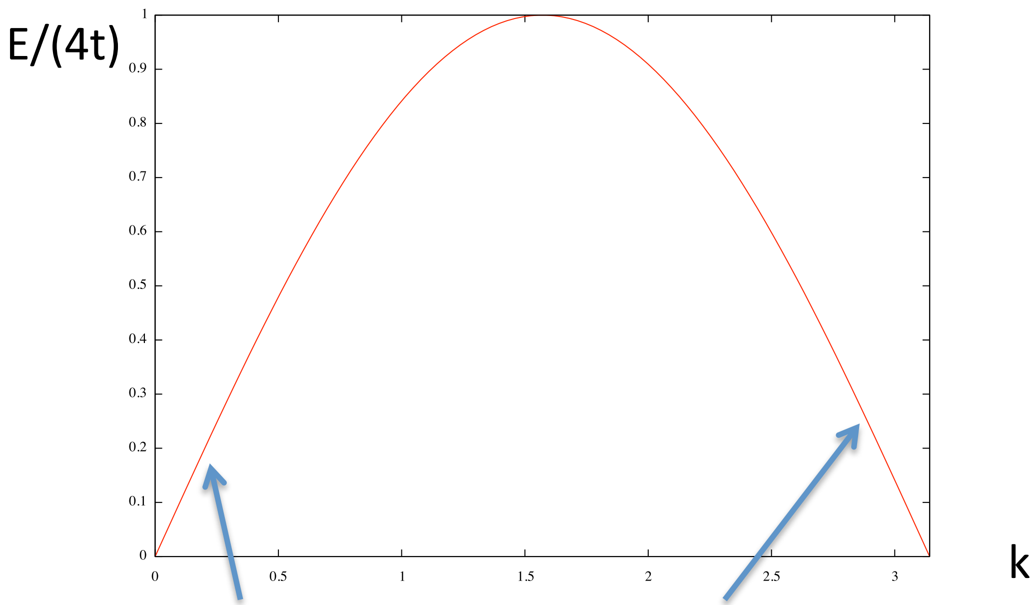
$$\gamma_j = \sqrt{\frac{2}{L}} \sum_{-\pi<k<\pi} e^{ikj}\gamma(k), \quad \{\gamma(k),\gamma(k')\} = \delta_{k,-k'}, \quad \gamma^+(k) = \gamma(-k)$$

$$H = 2t \sum_{-\pi<k<\pi} \gamma(-k)\gamma(k)\sin k.$$

We simply identify γ(k) as an annihilation operator for 0<k<π and as a creation operator for –π<k<0:

$$H = 4t \sum_{0<k<\pi} \gamma^+(k)\gamma(k)\sin k + E_0(k)$$

Annihilation and creation operators defined for k>0 only; low energy states near k=0 and k=π.



right-movers left-movers

$$\gamma_j/2 \approx \gamma_R(j) + (-1)^j \gamma_L(j),\ \{\gamma_R(x), \gamma_R(y)\} \approx (1/2)\delta(x-y),$$

$$H_0 \approx iv \int dx \left[\gamma_R \partial_x \gamma_R - \gamma_L \partial_x \gamma_L\right],\ v \equiv 4t$$

Here γ_R and γ_L are relativistic Majorana fermions.
Translation symmetry: $\gamma_R \to \gamma_R, \gamma_L \to -\gamma_L$
forbids a mass term: $im\gamma_R\gamma_L$. Corresponds to
Ising model at critical point.

To analyze effects of interaction term in H, we project it onto low energy subspace of relativistic Majorana fermions. Apart from renormalizing the velocity, these also produce an interaction term in the field theory:

$$H_{\text{int}} = 96g \int dx : \gamma_R \partial_x \gamma_R \gamma_L \partial_x \gamma_L :$$

Unlike in the Hubbard or spinless Dirac chain, interactions are irrelevant (dimension 4).
So, we expect the gapless non-interacting Majorana phase to persist up to a critical g for either sign of g.

III) Symmetry and Strong Coupling Phase

Will focus on g>0 here – SUSY occurs.
Unlike the Hubbard or spinless Dirac fermion model, the Majorana chain does not become trivial in the strong coupling limit, t=0.
To obtain a trivial model at t=0 it is convenient to explicitly break the translational symmetry – such symmetry breaking might occur spontaneously if not put in by hand:

$$H = \sum_j [it_1\gamma_{2j}\gamma_{2j+1} + it_2\gamma_{2j-1}\gamma_{2j}$$

$$+ g_1\gamma_{2j}\gamma_{2j+1}\gamma_{2j+2}\gamma_{2j+3} + g_2\gamma_{2j-1}\gamma_{2j}\gamma_{2j+1}\gamma_{2j+2}]$$

- Simplicity arises when $g_2=0$ (and $t_1=t_2=0$).
- Then it is convenient to combine every 2nd pair of Majorana's to make a Dirac:

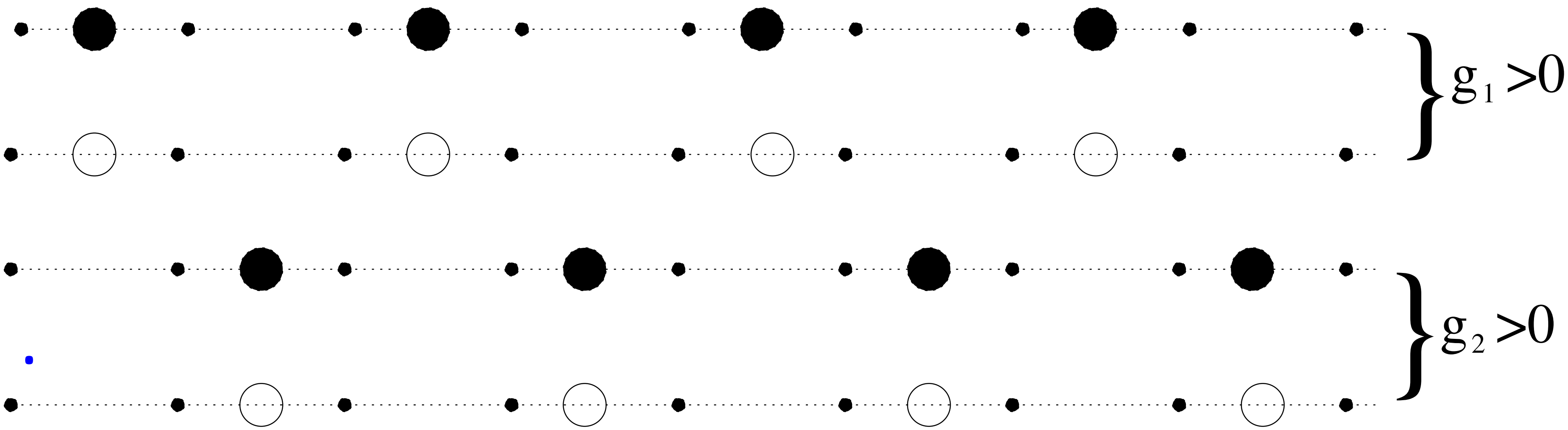
$$c_j \equiv (\gamma_{2j} + i\gamma_{2j+1})/2, \quad i\gamma_{2j}\gamma_{2j+1} = 2c_j^+c_j - 1 \equiv 2n_j - 1$$

$$H \rightarrow -g_1\sum_j (2n_j - 1)(2n_{j+1} - 1)$$

- For $g_1>0$ all (doubled) sites are filled or empty
- Ground state is 2-fold degenerate corresponding to a further spontaneous symmetry breaking.

Of course, if $g_1=t_1=t_2=0$ we can combine Majoranas to make Diracs on sites (2j-1) and 2j and get the same ground states translated by 1 site. The 2 ground states for only $g_1 \neq 0$ or only $g_2 \neq 0$ are sketched below.

Large filled circle means occupied Dirac level, empty circle means unoccupied Dirac level.



Similar states occur if only $t_1 \neq 0$:

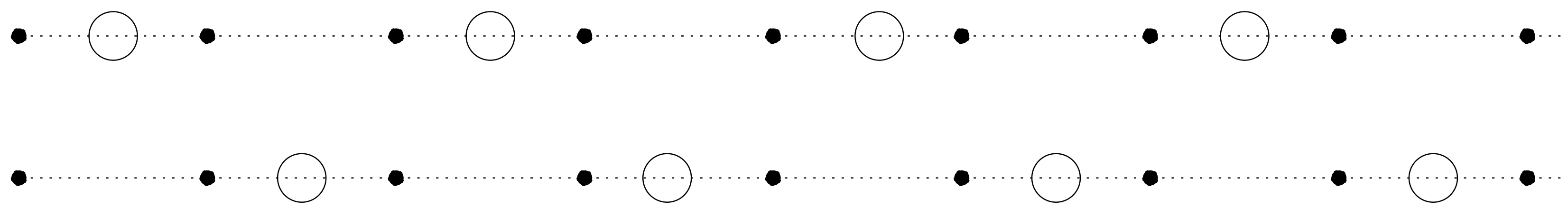
$$H = t_1 \sum_j (c_j^+ c_j - 1)$$

But now ground state is unique. Depending on sign of t_1, all states are filled or empty.

There is a particle-hole symmetry present when $t=0$: $\gamma_j \to (-1)^j \gamma_j,\ c_j \to c_j^+$

The ground states sketched above spontaneously break translation symmetry for case $t_1=t_2$, $g_1=g_2$.

For $t_1=t_2\neq 0$ and $g_1=g_2\neq 0$, we might expect a phase with spontaneously broken translational symmetry and 2 ground states.
For example, for t>0 the 2 ground states are:



That is, we form Dirac fermions either by combining γ_{2i} with γ_{2i+1} or by combining γ_{2i-1} with γ_{2i}. In either case all Dirac levels are empty. Ground states are only invariant under translation by 2 sites.

The full Hamiltonian can be written in terms of Dirac operators, useful for DMRG calculations:

$$H = \sum_j \{t[(2n_j - 1) + (-c_j^+ c_{j+1} + c_j c_{j+1} + h.c.)]$$

$$+g[-(2n_j - 1)(2n_{j+1} - 1) + (c_{j-1}^+ - c_{j-1})(2n_j - 1)(c_{j+1}^+ + c_{j+1})]\}$$

Particle number is not conserved,

2 types of pairing terms, 3 site interactions.

Particle number is only conserved mod 2.

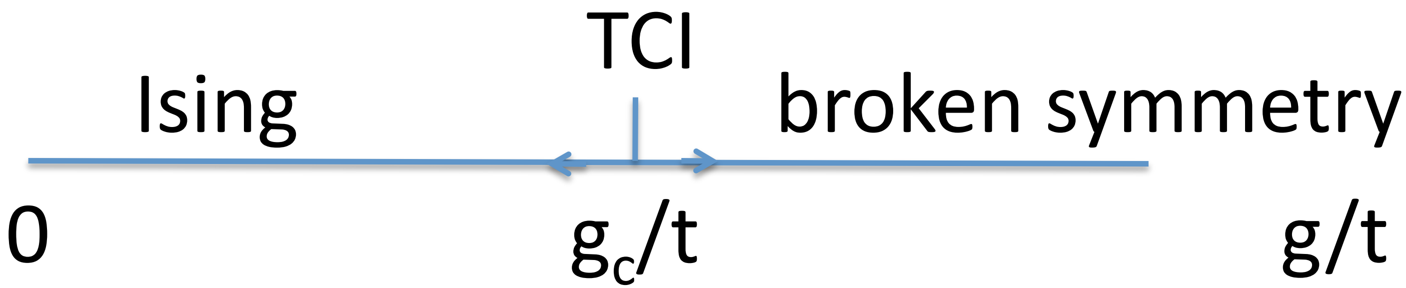
We define a fermion parity operator:

$$F = \prod_{j=0}^{L/2-1} (1 - 2n_j) = \prod_{j=0}^{L/2-1} (-i\gamma_{2j}\gamma_{2j+1}) = \pm 1$$

For L (even) Majorana sites: j=0,1,2, … (L-1)

As we go to large g, it is natural to expect a transition into the gapped phase with broken translational symmetry, allowing generation of a mass term. There are 2 ground states, corresponding to the 2 signs of the mass. It is also natural for the transition to the broken symmetry phase to be in the tri-critical Ising universality class. This is unique candidate critical theory with precisely 1 relevant operator allowed by symmetry. It is a c=7/10 conformal field theory whereas the free massless Majorana phase has c=1/2- Ising model.

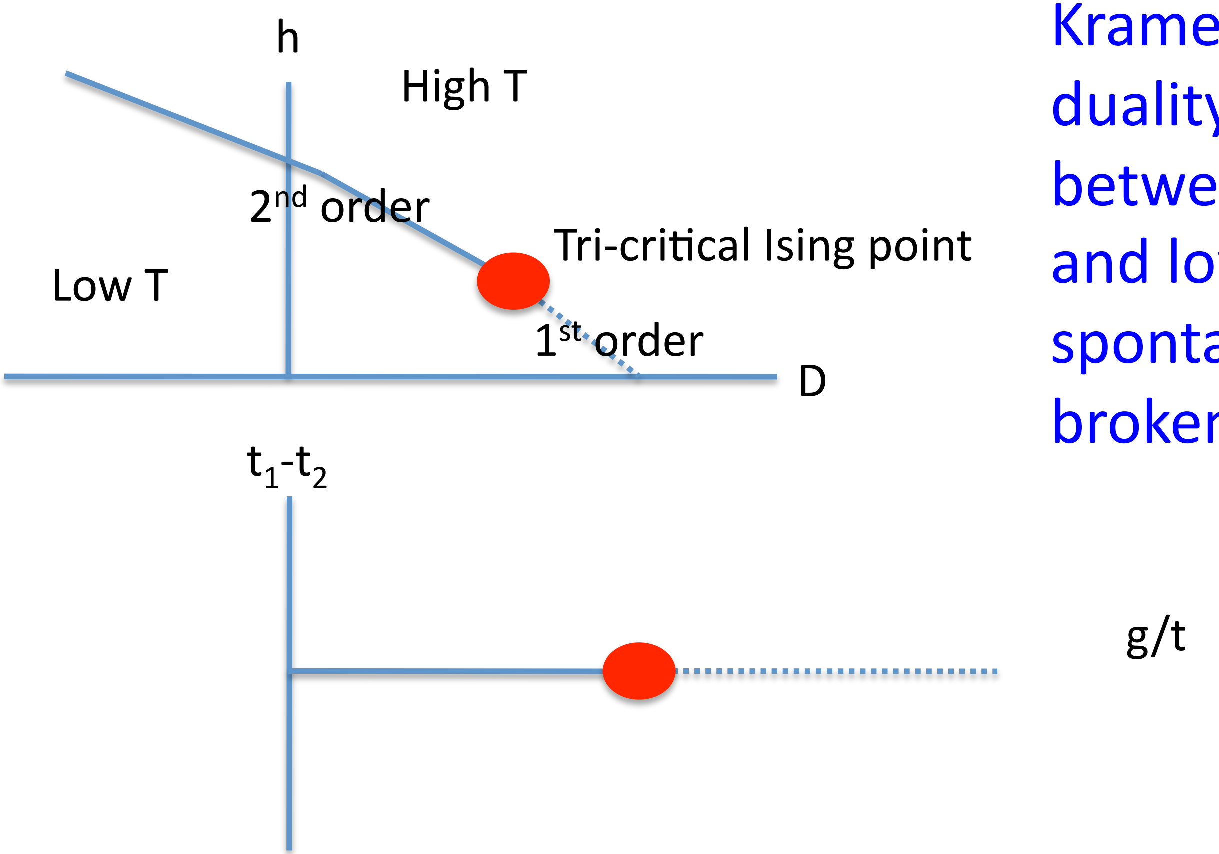
Purely fermionic model with irrelevant
interactions arises from integrating out boson
In SUSY model and can therefore be expected
to realize tri-critical Ising transition and SUSY
behavior:
Akulov and Volkov, 1973
Kastor, Martinec and Shenker, 1989



Note that the 2-fold degenerate ground state of the gapped phase for g>0 is consistent with this tri-critical Ising transition. The broken Z_2 symmetry is called Kramers-Wannier duality in Ising model. It forbids a mass term in the <u>Majorana fermion</u> representation.

Tri-critical Ising point known from diluted classical Ising model, equivalent to a spin-1 Ising chain:

$$H = \sum_j \left[-S_j^z S_{j+1}^z + h S_j^x + D\left(S_j^z\right)^2 \right]$$



Kramers-Wannier duality is a symmetry between high T and low T phases- spontaneously broken on 1st order line

Tri-critical Ising model is supersymmetric. (Friedan, Qiu and Shenker, 1985). The Neveu-Schwartz sector occurs for our model- Ising order parameter is non-local in fermions. chiral operators are

(I,ε,ε',ε''), dimension d=(0,1/10,3/5,3/2).

Boson fields:

(ε,ε)- energy operator in Ising model.

-odd under Kramers-Wannier Z_2 symmetry

(ε',ε') is even under Z_2, (ε'',ε'') odd

Fermions: $X_{R/L}$= (ε,ε'), (ε',ε), $X'_{R/L}$=(I,ε''), (ε'',I)

(conformal spin ½, 3/2).

Superfield:

$$\Phi = \varepsilon + \theta_R \chi_L + \theta_L \chi_R + \theta_R \theta_L \varepsilon'$$

Ginsburg-Landau Lagrangian density:

$$L = \frac{1}{4}(\overline{D}_\alpha \Phi)(D_\alpha \Phi) + W(\Phi),$$

$$W(\Phi) = \frac{1}{3}\Phi^3$$

The only relevant operator at the TCI point, allowed by Kramers-Wannier symmetry is ε', (sub-leading energy perturbation), preserves supersymmetry: $\delta W = (g_c - g)\Phi$

$V(\varphi)=(1/2)(\varphi^2+g_c-g)^2$. For $g<g_c$, minimum is at $\varphi=0$, $E_0>0$, $W''(0)=0$. SUSY is spontaneously broken, Majorana fermion is massless goldstino: Ising phase.

For $g>g_c$, minimum at $\varphi=\pm(g-g_c)^{1/2}$., $E_0=0$. SUSY unbroken but Z_2 spontaneously broken, fermion massive:

1st order region on Ising transition line.

Kastor, Martinec, Shenker, 1989

Establishing the tri-criticial Ising point with Density Matrix Renormalization Group was extremely challenging:

-it occurs at extremely large g/t≈250

-correlation length at g/t=∞ probably >1000

-we demonstrated TCI from finite size spectrum:

$$E_n = \varepsilon_0 L + \frac{2\pi v}{L}\left(-\frac{c}{12} + x_n\right)$$

-c is conformal charge (Ising:1/2, TCI: 7/10)

-fractions x_n are scaling dimensions of operators

-different (and known) for both Ising and TCI

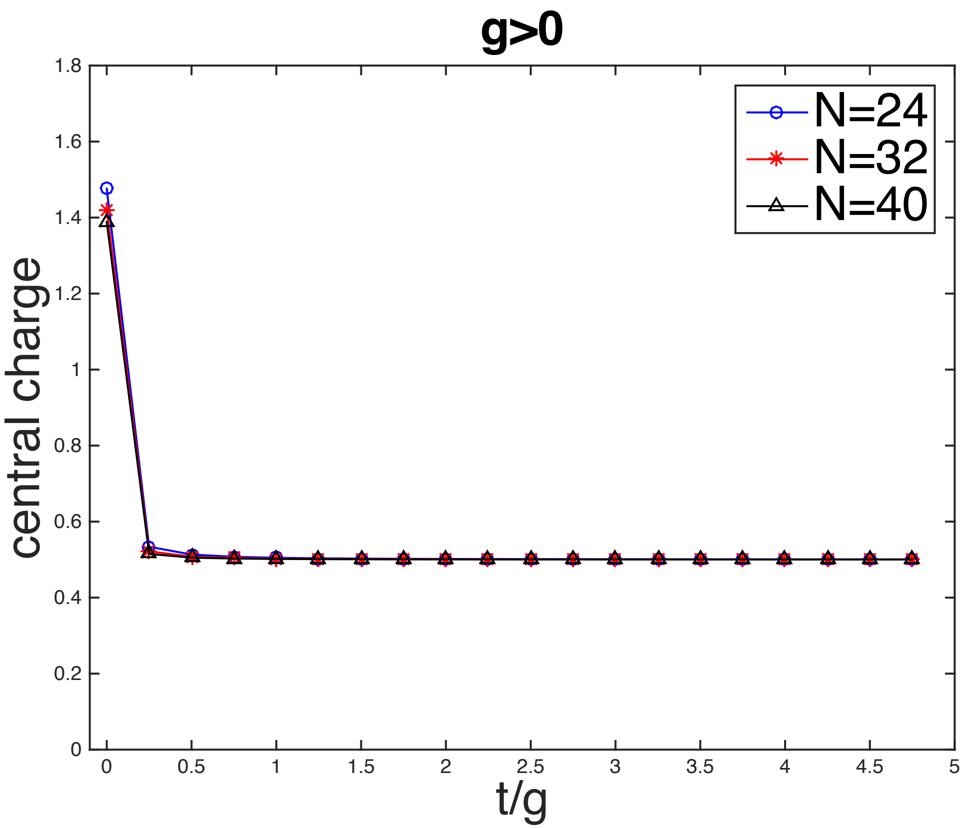
Finite size spectra with periodic or anti-periodic boundary conditions on the fermions are readily worked out for both Ising and TCI models. (Anti-periodic is modular invariant and corresponds to operator content.)

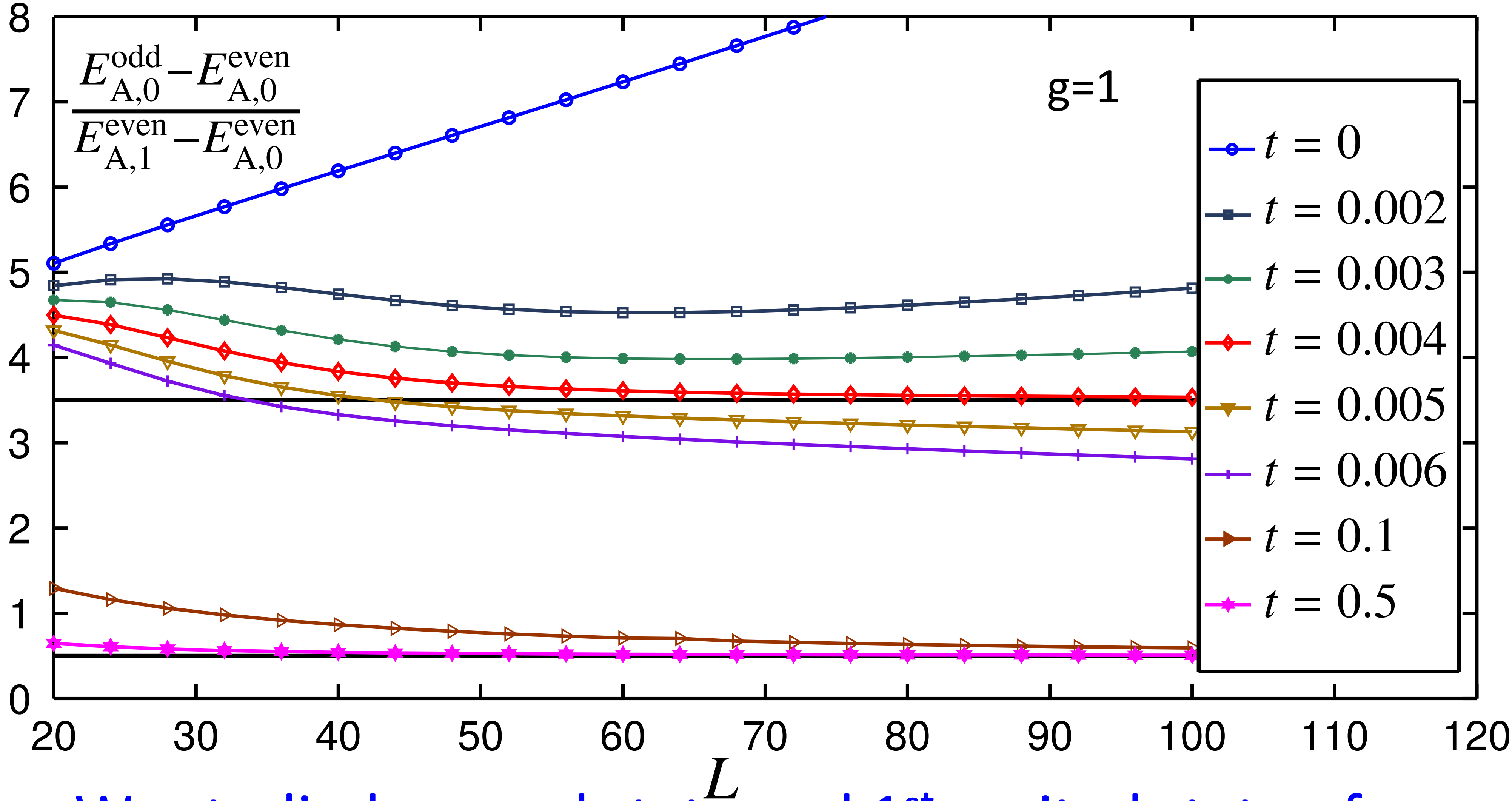
Ising: $Z_A=(X_I+X_\varepsilon)^2$, $Z_P=2(X_\sigma)^2$.

TCI: $Z_A=(X_\varepsilon+X_{\varepsilon'})^2+(X_I+X_{\varepsilon''})^2$, $Z_P=2(X_\sigma)^2+2(X_{\sigma'})^2$

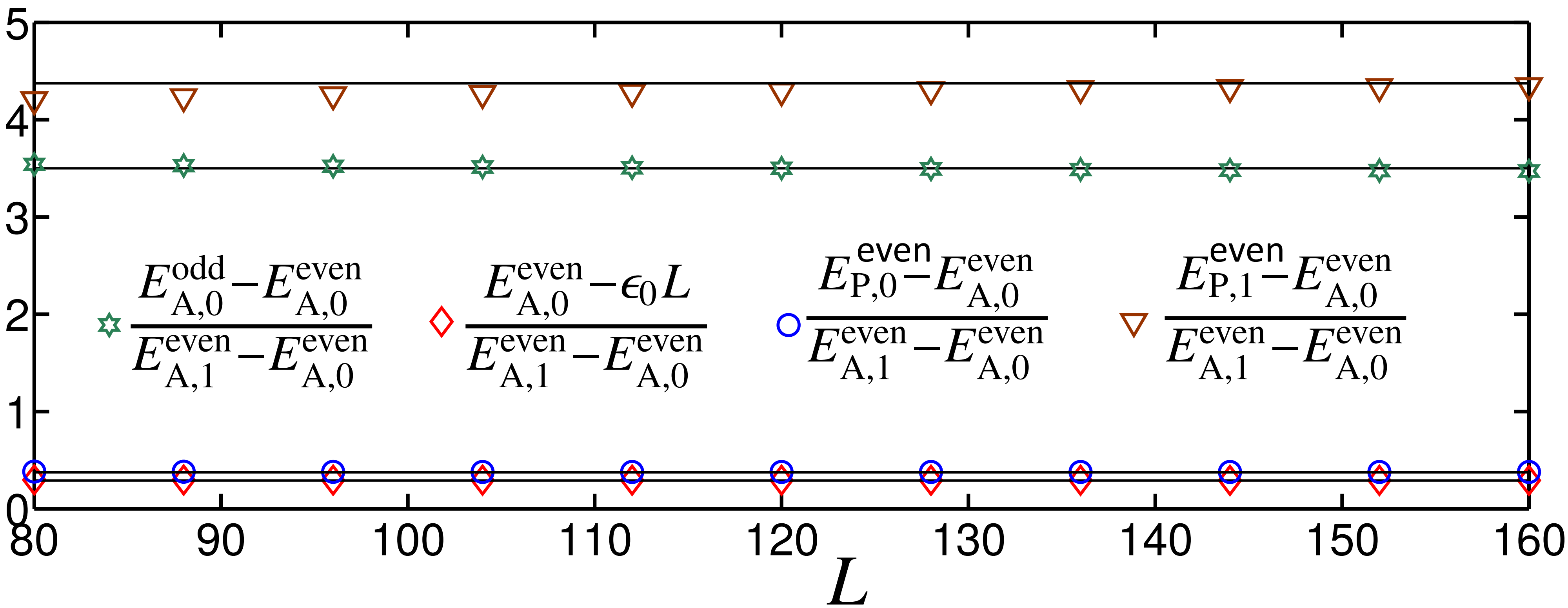
Here X labels conformal towers, σ is order parameter, σ' sub-leading order parameter in TCI. We can also readily identify fermion parity of all states.

c can also be measured from entanglement entropy:

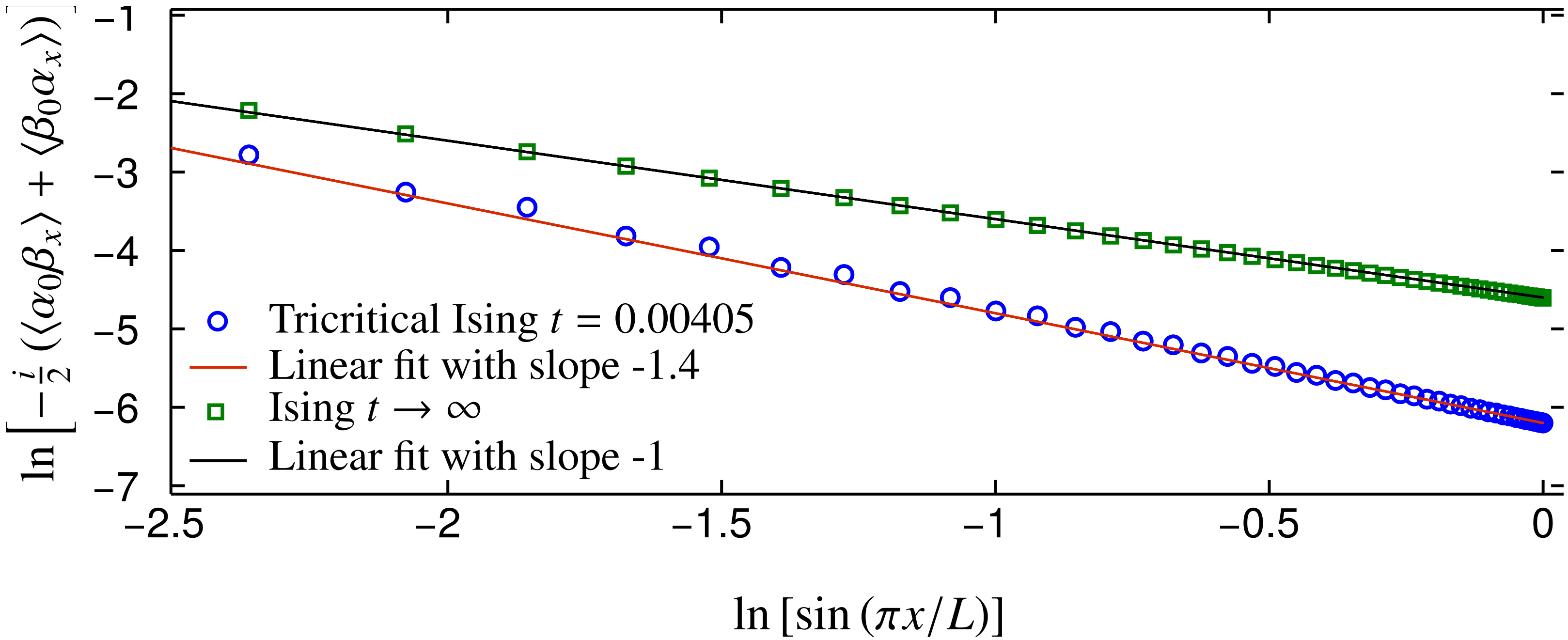




- We studied ground state and 1st excited state of even and odd fermion parity with anti-periodic B.C.'s
- ratio=1/2 for Ising, 7/2 for TCI



Also did DMRG for periodic boundary conditions.
Scaling dimensions x=7/24, 3/8, 7/2, 35/8
agree well with DMRG data at t/g=.00405



Fermion correlation function has power-law decay (for infinite length chain)

-exponent η=1 in Ising phase, η=7/5 in TCI phase

Scanning Tunnelling Microscopy Signatures of Majoranas

Voltage dependence of tunneling rate determined by Fourier transform of retarded Green's function $\langle \gamma_j(t)\gamma_j(0)\rangle$. In Ising phase, ~1/t corresponding to a constant density of states; I~V. At TCI point, assuming low frequency Green's function is dominated by low energy excitations, $I \sim |V|^{7/5}$ Related equal time correlation function decays as 1/x in Ising phase and $1/|x|^{7/5}$ at TCI- agrees with DMRG as I showed earlier.

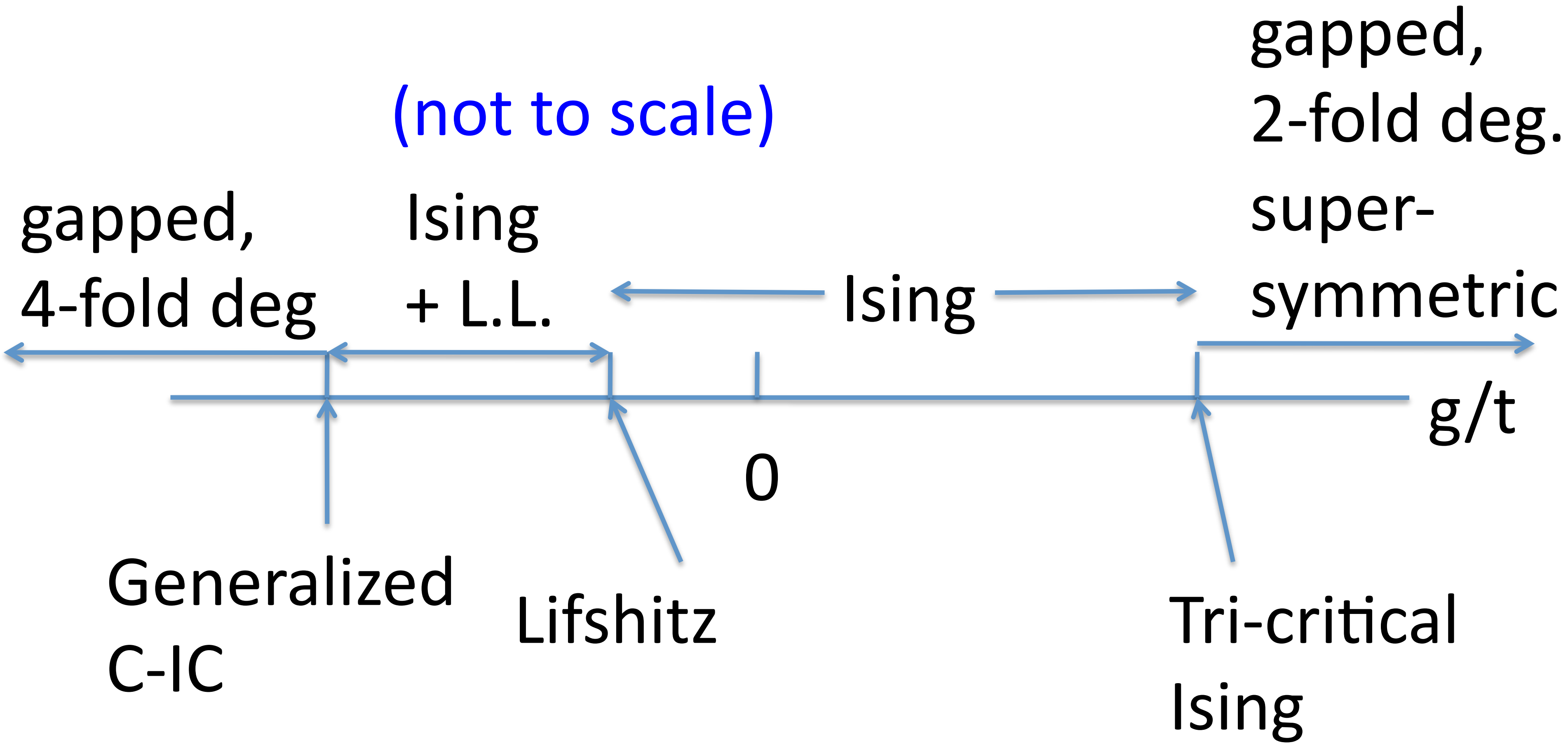
For $g>g_c$ there is a gap in density of states. SUSY implies same gap for fermionic excitation (STM) as bosonic excitations (Cooper pair tunnelling or particle-hole excitations induced by photon absorption).

How should we think about gapped phase with broken Z_2 symmetry and unbroken SUSY?

Basic excitations are solitons between 2 ordered phases. Apparently, there are no bound states:
A. Zamalodchikov, 1989
Lassig, Mussardo, Cardy, 1991

Soliton has Majorana mode at core.
2 Majorana operators can be combined to make a normal creation operator so a soliton-antisoliton pair can be bosonic or fermionic. SUSY implies they have exactly the same energy. Difficult to detect because soliton-antisoliton interaction drops off exponentially with separation and they don't form boundstates.

Complete phase diagram



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