1. IAS Summer Collaborators Report

This is a short report of the results obtained by Jack Jeffries, Devlin Mallory, Claudia Miller, Josh Pollitz, and Eamon Quinlan–Gallego during our two week stay at the IAS as part of the Summer Collaborators program. The proposed project aimed to better understand a notion of derived differential operators defined in [Yan21]; below, we detail some of our major accomplishments during the program.

Throughout assume S is a smooth algebra over a fixed base field, and set $R = S/I$ for some ideal $I \subset S$. Let $\mathcal{D}_S$ denote the ring differential operators on S, and set

$$\mathcal{E}_R :- \mathsf{RHom}_{\mathcal{D}_S}(\mathcal{D}_S/I\mathcal{D}_S, \mathcal{D}_S/I\mathcal{D}_S)\,,$$

the dg algebra of derived differential operators on R discussed in [Yan21].

Also, for a dg algebra A we let $\mathsf{D}(A)$ denote its derived category of right dg A-modules and write $\mathsf{Perf}(A)$ for its full subcategory of the perfect dg A-modules. When there is a map $S \to A$ we also consider the subcategories $\mathsf{D}_I(A)$ and $\mathsf{Perf}_I(A)$ of the previously defined categories whose objects are those whose cohomology is I-power torsion (when regarded as an S-module via restriction of scalars along $S \to \mathrm{H}^0(A)$).

1.1. Correcting the setting from [Yan21].

During a visit to SLMath in Spring 2024, a discussion with Travis Schedler confirmed that there are several foundational gaps in [Yan21]. Our first result clarified the situation, by providing a markedly different proof to a more general result using a derived Morita equivalence from [DG02].

Theorem 1.1. *With the notation above, there are equivalences of triangulated categories*

$$\begin{array}{ccc} \mathsf{D}_I(\mathcal{D}_S) & \xrightarrow{\quad\equiv\quad} & \mathsf{D}(\mathcal{E}_R) \\ \subset\uparrow & & \subset\uparrow \\ \mathsf{D}^b_I(\operatorname{mod}\mathcal{D}_S) = \mathsf{Perf}_I(\mathcal{D}_S) & \xrightarrow{\equiv} & \mathsf{Perf}(\mathcal{E}_R). \end{array}$$

As opposed to [Yan21], our result has no assumption on the characteristic of the base field, and also leads to a natural generalization to the scheme theoretic setting. Furthermore, our techniques were rather elementary (only involving the derived Morita equivalence from [DG02]) and avoiding the machinery of ∞-categories and higher algebra in [GR14]. One of the questions we hope to pursue in the future is: When is there a strong generator of $\mathsf{Perf}(\mathcal{E}_R)$? We know this is the case when R is smooth or cuspidal.

1.2. Positive characteristic.

In this section we assume the algebra R has prime characteristic $p > 0$. In this case, R has a Frobenius endomorphism $F\colon R \to R$ and it is well known that the classical/affine ring of differential operators on R has a description in terms iterates of this endomorphism. One of the main results of the program is extending the classical situation to providing an analogous (derived) characterization of the dg-algebra $\mathcal{E}_R$ in terms of iterates of the Frobenius.

Given an integer $e \geqslant 0$, we let $R_e = R$ and we view R as an R_e-algebra through the e-th iterate of the Frobenius $F^e\colon R_e \to R$. Note that when R is reduced the morphism F^e identifies R_e with the subring R^{p^e} of p^e-powers of R. With this notation, our description of $\mathcal{E}_R$ is listed below:

Theorem 1.2. *With the notation set above, if R is further assumed to have prime characteristic $p > 0$, then*

$$\mathcal{E}_R \simeq \operatorname*{colim}_e \mathsf{RHom}_{R_e}(R, R).$$

More generally, the derived functor of differential operators of [SVdB97] *can be calculated similarly when M, N are finitely generated R-modules:*

$$\mathcal{E}_R(M,N) := \operatorname*{colim}_n \mathsf{RHom}_R(P^n \otimes_R M, N) = \operatorname*{colim}_e \mathsf{RHom}_{R_e}(M,N). \tag{1}$$

The proof goes roughly as follows. There is a natural map between the two sides of (1), and so it suffices to show the induced map on cohomology is an isomorphism; that is to say,

$$\operatorname*{colim}_n \operatorname{Ext}^i_R(P^n_R \otimes_R M, N) \cong \operatorname*{colim}_e \operatorname{Ext}^i_{R_e}(M,N)$$

for all $i \geqslant 0$. By Matlis duality this reduces to a statement on maps of Tor-modules. The relevant theorem we show is the following.

Theorem 1.3. *With the setup of Theorem 1.2, if M and N are finitely generated R-modules and $i > 0$, then we have the following isomorphism*

$$\lim_e \operatorname{Tor}^{R_e}_i(M,N) = 0\,.$$

The proof of Theorem 1.3 involves simplicial methods and an application of the Artin–Rees lemma. We suspect this result to be of independent interest to commutative algebraists studying singularities in positive characteristic; namely, the vanishing of the maps in the system in Theorem 1.3 is subtle. We have simple examples which show that transition maps in the pro-system $\{\operatorname{Tor}^{R_e}_i(M,N)\}_{e=0}^{\infty}$ need not be zero (and even have examples witnessing nonzero compositions for an arbitrarily long uniform length). In contrast, there is the following strong vanishing one gets in the local case, when M and N are the residue field (this recovers a classical result of André).

Theorem 1.4. *Let R be a local ring of prime characteristic $p > 0$ with residue field k. Then for all $e, i > 0$ the following induced morphisms are zero:*

$$\operatorname{Tor}^{R_e}_i(k,k) \to \operatorname{Tor}^{R_{e-1}}_i(k,k)\,.$$

1.3. **Examples, $\mathcal{D}_R$-module structures, and finiteness conditions.** Recall that, with the notation as in the introduction, the ring $\mathrm{H}^0(\mathcal{E}_R)$ is the classical ring of differential operators $\mathcal{D}_R$ on R as defined in [Gro65]. The higher cohomology modules $H^i(\mathcal{E}_R)$ are naturally $(\mathcal{D}_R, \mathcal{D}_R)$-bimodules; one of the goals of the collaboration was to better understand these bimodules.

In the case when $R = S/fS$ is a hypersurface ring all higher cohomologies of $\mathcal{E}_R$ vanish except for $\mathrm{H}^1(\mathcal{E}_R)$. We use a description of $\mathrm{H}^1(\mathcal{E}_R)$ given in [JS23]:

$$\mathrm{H}^1(\mathcal{E}_R) \cong H^n_\Delta(R \otimes_k \omega_R), \tag{2}$$

where $n :- \dim(S) = \dim(R) + 1$, where $\omega_R = \omega_S/\omega_S f$ denotes the canonical module of R, and $\Delta \subset R \otimes_k R$ denotes the diagonal ideal. In loc. cit. it is shown that the above is an isomorphism of left $\mathcal{D}_R$-modules. Nonetheless, ω_R has a right $\mathcal{D}_R$-module structure, which induces a right $\mathcal{D}_R$-module structure on the right-hand side of (2), and we show that (2) is also an isomorphism of $(\mathcal{D}_R, \mathcal{D}_R)$-bimodules.

Understanding these bimodule structures allows us to analyze the $(\mathcal{D}_R, \mathcal{D}_R)$-bimodule structure of $\mathrm{H}^1(\mathcal{E}_R)$ over some hypersurface singularities, and in particular its finite generation. We start by a further analysis of an example from [Yan21]:

Example 1.5. Suppose $\mathrm{char}(k) = 0$, and let $R :- k[x, y]/(xy)$. Then $\mathrm{H}^1(\mathcal{E}_R)$ is finitely generated as a $(\mathcal{D}_R, \mathcal{D}_R)$-bimodule, but not as a left or right $\mathcal{D}_R$-module.

We also consider a simple ring of invariants:

Example 1.6. Let $R = k[x^2, xy, y^2] \cong k[a, b, c]/(ac - b^2)$. Then $\mathrm{H}^1(\mathcal{E}_R)$ is not finitely generated as a left $\mathcal{D}_R$-module.

In Example 1.6, viewing the right-hand side of (2) as the $(\mathbb{Z}/2)$-invariants of the analogous local cohomology module of $k[x, y]$, and assuming certain compatibility relations of the $(\mathcal{D}_R, \mathcal{D}_R)$-bimodule structures involved, we are able to show that $\mathrm{H}^1(\mathcal{E}_R)$ would also fail to be finitely generated as a right $\mathcal{D}_R$-module, but that it would be finitely generated as a bimodule.

The following example shows that $\mathrm{H}^1(\mathcal{E}_R)$ may fail to be finitely generated even as a $(\mathcal{D}_R, \mathcal{D}_R)$-bimodule.

Example 1.7. Let $R = \mathbb{C}[x_1, \ldots, x_n]/(x_1^d + g(x_2, \ldots, x_n))$, where $\deg g = d \geqslant 3$ and $n \geqslant 4$, with isolated singularity at the origin. Then $\mathcal{D}_R$ has no elements of negative degree, but $\mathrm{H}^1(\mathcal{E}_R)$ has elements of arbitrarily negative degree, and thus cannot be finitely generated as a bimodule over $\mathcal{D}_R$.

We believe this can be generalized to an arbitrary homogeneous polynomial of degree $d \geqslant 3$ with isolated singularity; this is based on a sketch involving more geometric arguments during the program.

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