PCMI Summer School
problems with Galois cohomology

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1 Quadratic forms

Definition 1.1. *Let F be a field. We define the fundamental ideal $I(F) \subset W(F)$ to the classes of quadratic forms of even dimension. We let $I^n(F) = (I(F))^n$.*

Definition 1.2. *For $a \in F^*$, let $\langle\langle a \rangle\rangle$ denote the quadratic form $\langle 1, -a \rangle$. For $a_1, \ldots, a_n \in F^*$, let $\langle\langle a_1, \ldots, p_n \rangle\rangle = \langle\langle a_1 \rangle\rangle \otimes \cdots \otimes \langle\langle a_n \rangle\rangle$. We call such forms n-fold Pfister forms.*

Exercise 1.3. *Show that $I^n(F)$ is generated by the classes of n-fold Pfister forms.*

Exercise 1.4. *By Graham-Schmidt, if q is a quadratic form on a vector space V and $v \in V$, we can write $V = Fv \perp W$ for some complementary subspace W. Show that if V is 2 dimensional with $q = \langle a, b \rangle$ and if $v \in V$ with $q(v) = c \in F^*$, then we may write $q = \langle c, abc \rangle$.*

As a hint for the above exercise, consider the determinant of the Graham matrix of the bilinear form associated to q.

Exercise 1.5. *Show that $\langle\langle a, 1-a \rangle\rangle$ is hyperbolic.*

Exercise 1.6. *Show that the map $\widetilde{f} : F^* \times \cdots F^* \to W(F)$ sending $(a_1, \ldots, a_n)$ to $\langle\langle a_1, \ldots, a_n \rangle\rangle$ satisfies $\widetilde{f}(a_1, \ldots, a_n) = 0$ whenever $a_i + a_j = 1$ for some i, j.*

Exercise 1.7. *Show that $\widetilde{f}$ induces a homomorphism $f : K^M_\bullet(F) \to gr^I_\bullet W(F)$ of graded rings.*

Exercise 1.8. *Show that $f(2\, K^M_\bullet(F)) = 0$.*

Definition 1.9. *For a field F, the n'th Pfister number of F, denoted $\mathrm{Pf}_n(F)$, is the minimum number m such that every element of $I^n(F)$ can be written as a sum (or difference/integral linear combination) of at most m n-fold Pfister forms.*

Exercise 1.10. *Show that the Pfister form $\langle\langle 1, a_2, \ldots, a_n \rangle\rangle$ is always isotropic.*

We will use the following helpful fact: if $\phi = \langle\langle a_1, \ldots, a_n \rangle\rangle$ and if $\phi(x) = b$ has a solution, then we may write $\phi \cong \langle\langle -b, b_2, \ldots, b_n \rangle\rangle$.

Exercise 1.11. *Show that if ϕ is a Pfister form and $\phi(x) = -1$ has a solution then ϕ is isotropic.*

Exercise 1.12. *Use the previous exercises to show that if a pfister form is isotropic, then it is hyperbolic.*

Exercise 1.13. *Show that if ϕ is a Pfister form and ψ is a subform of dimension greater than half the dimension of ϕ, then ϕ is isotropic if and only if ψ is isotropic.*

2 Central simple algebras and Brauer groups

Definition 2.1 (quaternion algebras)**.** *For a field F of characteristic not 2, and elements $a, b \in F^*$, define the associative algebra $(a, b)_{-1}$ to be the algebra generated by elements u, v with the relations $u^2 = a, v^2 = b, uv = -vu$.*

Exercise 2.2. *Show that $(a, b)_{-1}$ is a division algebra if and only if the form $\langle\langle a, b\rangle\rangle$ is anisotropic.*

Exercise 2.3 (not so easy without ingenuity)**.** *Show that if $(a, b)_{-1}$ is not division then it is isomorphic to $M_2(F)$.*

3 Galois cohomology

Suppose H is a functor from fields to torsion Abelian groups. For E/F, denote the map $H(F) \to H(E)$ by $res_{E/F}$, and suppose we also have maps $cor_{E/F} : H(E) \to H(F)$ such that $cor_{E/F} res_{E/F}$ is multiplication by $[E : F]$ when E/F is a finite extension.

As with Galois cohomology, for $\alpha \in H(F)$, define $ind(\alpha) = gcd\{[E : F] \mid res_{E/F}\alpha = 0\}$ and let $per(\alpha)$ be the order of α in $H(F)$.

Exercise 3.1. *Show that if $H(\overline{F}) = 0$ for $\overline{F}$ an algebraically closed field, that $per(\alpha)|ind(\alpha)$ for $\alpha \in H(F)$ and that $per(\alpha)$ and $ind(\alpha)$ have the same prime divisors.*

Recall that $ssd_\ell^{n,m}(F)$ is the minimum d such that $ind(\alpha)|per(\alpha)^d$ for any $\alpha \in H^n(E, \mu_\ell^{\otimes m})$ where E/F is any finite extension.

Exercise 3.2. *Show that $ssd_{\ell_1\ell_2}^{n,m}(F)$ is the max of $ssd_{\ell_i}^{n,m}(F)$ for $i = 1, 2$ if $(\ell_1, \ell_2) = 1$.*

Exercise 3.3. *Show that $ssd_p^{n,m}(F)$ is independent of m if p is prime.*

Exercise 3.4. *Show that $ssd_{p^n}^{n,m}(F) \leq ssd_p^{n,m}(F)$.*