TORSORS OVER AFFINE CURVES, PCMI, JULY 2024// PRELIMINARY VERSION

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Abstract. The lectures are an introduction to torsors in Algebraic Geometry with special attention to the case of affine algebraic curves.

Contents

1. Introduction

The theory of fibrations and principal fibrations is ubiquous in Topology and Differential Geometry In 1955, Grothendieck investigated a general theory of fibrations focusing on functoriality issues [?]. In 1958, Grothendieck and Serre extended the setting of G–bundles in algebraic geometry by means of the étale topology [?].

For simplicity we shall present this theory over rings or equivalently over affine schemes. The general framework is close to that and can be found in other references [?, ?, ?, ?].

We shall focus on the case of an affine smooth curve over a field, starting with vector bundles and quadratic vector bundles. Important cases are the affine line and the affine punctured line.

For further topics, we recommend the survey *Problems about torsors over regular rings* of K. Česnavičius [?].

2. The Swan-Serre correspondence

This is the correspondence between locally free modules of finite rank and vector bundles, it arises from the case of a paracompact topological space [?].

We explicit it in the setting of affine schemes following the book of Görtz-Wedhorn [?, ch. 11] up to slightly different conventions.

Date: July 5, 2024.

The author was supported by the project "Group schemes, root systems, and related representations" founded by the European Union - NextGenerationEU through Romania's National Recovery and Resilience Plan (PNRR) call no. PNRR-III-C9-2023- I8, Project CF159/31.07.2023, and coordinated by the Ministry of Research, Innovation and Digitalization (MCID) of Romania.

2.1. **Vector group schemes.** Let R be a ring (commutative, unital). The additive R–group scheme is $\mathbb{G}_{a,R} = \mathrm{Spec}(R[t])$ and is part of a wider family.

(a) Let M be an R–module. We denote by $\mathbf{V}(M)$ the affine R–scheme defined by $\mathbf{V}(M) = \mathrm{Spec}\big(\mathrm{Sym}^\bullet(M)\big)$; it is affine over R and represents the R–functor $S \mapsto \mathrm{Hom}_S(M \otimes_R S, S) = \mathrm{Hom}_R(M, S)$ [**?**, 9.4.9].

It is called the *vector group scheme* attached to M, this construction commutes with arbitrary base change of rings $R \to R'$. We have $\mathbf{V}(R) = \mathbb{A}^1_R = \mathrm{Spec}(R[t])$, that is, the affine line over R.

Proposition 2.1. [**?**, I.4.6.1] *The functor $M \to \mathbf{V}(M)$ induces an anti-equivalence of categories between the category of R–modules and that of vector group schemes over R. An inverse functor is $\mathfrak{G} \mapsto \mathfrak{G}(R)$.*

(b) We assume now that M is locally free of finite rank and denote by $M^\vee$ its dual. In this case $\mathrm{Sym}^\bullet(M)$ is of finite presentation [**?**, 9.4.11]. Also the R–functor $S \mapsto M \otimes_R S$ is representable by the affine R–scheme $\mathbf{V}(M^\vee)$ which is also denoted by $\mathbf{W}(M)$ [**?**, I.4.6].

Remark 2.2. Romagny has shown that the finite locally freeness condition on M is a necessary condition for the representability of $\mathbf{W}(M)$ by a group scheme [**?**, th. 5.4.5].

Let $r \geq 0$ be an integer.

Definition 2.3. *A vector bundle of rank r over* $\mathrm{Spec}(R)$ *is an affine R-scheme X such that there exists a partition $1 = f_1 + \cdots + f_n$ and isomorphisms $\phi_i : \mathbf{V}((R_{f_i})^r) \xrightarrow{\sim} X \times_R R_{f_i}$ such that $\phi_i^{-1}\phi_j : \mathbf{V}((R_{f_i f_j})^r) \xrightarrow{\sim} \mathbf{V}\big((R_{f_i f_j})^r\big)$ is a linear automorphism of $\mathbf{V}\big((R_{f_i f_j})^r\big)$ for $i, j = 1, \dots, n$.*

Theorem 2.4. *(Swan-Serre's correspondence) The above functor $M \mapsto \mathbf{V}(M)$ induces an anti-equivalence of categories between the groupoid of locally free R–modules of rank r and the groupoid of vector bundles over* $\mathrm{Spec}(R)$ *of rank r.*

Proof. See [**?**, prop. 11.7] for the general case (i.e. over a base scheme). We check first that the functor is well-defined. If M is locally free of rank r, there exists a partition $1 = f_1 + \cdots + f_n$ and trivializations $\psi_i : (R_{f_i})^r \xrightarrow{\sim} M_{f_i}$. It follows that the maps $(\psi_i)^{-1}\psi_j : (R_{f_i f_j})^r \xrightarrow{\sim} (R_{f_i f_j})^r$ is a linear isomorphism for $i, j = 1, \dots, n$. By applying the functor $\mathbf{V}$, we get that $\mathbf{V}(M)$ is a vector bundle of rank r and the trivializations are the

$$\phi_i = (\psi_i^{-1})^* : \mathbf{V}((R_{f_i})^r) \xrightarrow{\sim} \mathbf{V}(M) \times_R R_{f_i}.$$

It follows that $\mathbf{V}$ is well-defined and is fully faithful. To check it is essentially surjective, it is enough to observe that the inverse functor $\mathfrak{G} \to \mathfrak{G}(R)$ of $\mathbf{V}$ applies a vector bundle of rank r to a locally free R–module of rank r. $\square$

Examples 2.1.1. (a) Given a smooth map of affine schemes $X = \mathrm{Spec}(S) \to Y = \mathrm{Spec}(R)$ of relative dimension $r \geq 1$, the tangent bundle $T_{X/Y} = \mathbf{V}(\Omega^1_{S/R})$ is a vector bundle over $\mathrm{Spec}(S)$ of dimension r [**?**, 16.5.12].

(b) The tangent bundle of the real sphere $Z = \mathrm{Spec}\Big(\mathbb{R}[x,y,z]/(x^2+y^2+z^2-1)\Big)$ is an example of a vector bundle of dimension 2 which is not trivial. It can be proven by differential topology (hairy ball theorem) but there are also algebraic proofs, see for instance [**?**]. A consequence is that Z cannot be equipped with a structure of a real algebraic group.

(c) Note that this tangent bundle extended to $\mathbb{C}$ becomes free. This is consequence of Murty-Swan's theorem [**?**] since it is stably free.

2.2. **Linear groups.** Let M be a locally free R–module of finite rank. We consider the R–algebra $\mathrm{End}_R(M) = M^\vee \otimes_R M$. It is a locally free R–module of finite rank so that we can consider the vector R–group scheme $\mathbf{V}\big(\mathrm{End}_R(M)\big)$ which is an R–functor with values in associative and unital algebras [**?**, 9.6.2]. Now we consider the R–functor $S \mapsto \mathrm{Aut}_S(M \otimes_R S)$. It is representable by an open R–subscheme of $\mathbf{W}\big(\mathrm{End}_R(M)\big)$ which is denoted by $\mathrm{GL}(M)$ (*loc. cit.*, 9.6.4). We bear in mind that the action of the group scheme $\mathrm{GL}(M)$ on $\mathbf{W}(M)$ (resp. $\mathbf{V}(M)$) is a left (resp. right) action.

In particular, we denote by $\mathrm{GL}_r = \mathrm{Aut}(R^r)$.

Remark 2.5. For R noetherian, Nitsure has shown that the finite locally freeness condition on M is a necessary condition for the representability of $\mathrm{GL}(M)$ by a group scheme [**?**].

(c) If $\mathcal{B}$ is a locally free R–algebra of finite rank, we recall that the functor of invertible elements of $\mathcal{B}$ is representable by an affine R-group scheme which is a principal open subset of $\mathbf{W}(\mathcal{B})$. It is denoted by $\mathrm{GL}_1(\mathcal{B})$ [**?**, 2.4.2.1].

2.3. **Cocycles.** Let M be a locally free R–module of rank r. There exists a partition $1 = f_1 + \cdots + f_n$ of R and isomorphisms $\phi_i : (R_{f_i})^r \xrightarrow{\sim} M \times_R R_{f_i}$. Then the $R_{f_i f_j}$–isomorphism $\phi_i^{-1}\phi_j : (R_{f_i f_j})^r \xrightarrow{\sim} (R_{f_i f_j})^r$ is linear so defines an element $g_{i,j} \in \mathrm{GL}_r(R_{f_i f_j})$. More precisely we have $(\phi_i^{-1}\phi_j)(v) = g_{i,j} \,.\, v$ for each $v \in (R_{f_i f_j})^r$ (in other words, $(R_{f_i f_j})^r$ is seen as column vectors).

Lemma 2.6. *The element $g = (g_{i,j})$ is a 1–cocycle, that is, satisfies the relation*

$$g_{i,j}\, g_{j,k} = g_{i,k} \in \mathrm{GL}_r(R_{f_i f_j f_k})$$

for all $i, j, k = 1, \dots, n$.

Proof. Over $R_{f_i f_j f_k}$ we have $\phi_i^{-1}\phi_k = (\phi_i^{-1}\phi_j) \circ (\phi_j^{-1}\phi_k) = L_{g_{i,j}} \circ L_{g_{j,k}} = L_{g_{i,j} g_{j,k}}$ where L stands for the left translation on GL_r. □

If we replace the ϕ_i's by the $\phi'_i = \phi_i \circ g_i$ for $g_i \in \mathrm{GL}_r(R_{f_i})$, we get $g'_{i,j} = g_i^{-1} g_{i,j} g_j$ and we say that $(g'_{i,j})$ is cohomologous to $(g_{i,j})$.

We denote by $\mathcal{U} = \big(\mathrm{Spec}(R_{f_i}\big)_{i=1,..,n}$ the affine cover of $\mathrm{Spec}(R)$, by $Z^1(\mathcal{U}/R, \mathrm{GL}_r)$ the set of 1-cocycles and by $H^1(\mathcal{U}/R, \mathrm{GL}_r) = Z^1(\mathcal{U}/R, \mathrm{GL}_r)/\sim$ the set of 1–cocycles modulo the cohomology relation. The set $H^1(\mathcal{U}/R, \mathrm{GL}_r)$ is called the pointed set of Čech cohomology with respect to $\mathcal{U}$.

Summarizing we attached to the vector bundle $\mathbf{V}(M)$ of rank r a class $\gamma(M) \in H^1(\mathcal{U}/R, \mathrm{GL}_r)$.

Conversely by Zariski glueing, we can attach to a cocycle $(g_{i,j})$ a vector bundle $\mathbf{V}_g$ over R of rank r equipped with trivializations $\phi_i : \mathbf{V}(R_{f_i}^r) \xrightarrow{\sim} \mathbf{V}_g \times_R R_{f_i}$ such that $\phi_i^{-1}\phi_j = g_{i,j}$.

Lemma 2.7. *The pointed set $H^1(\mathcal{U}/R, \mathrm{GL}_r)$ classifies the isomorphism classes of vector bundles of rank r over $\mathrm{Spec}(R)$ which are trivialized by $\mathcal{U}$.*

For the proof, see [**?**, 11.15]. We can pass this construction to the limit over all affine open subsets of X. We define the pointed set $\check{H}^1_{Zar}(R, \mathrm{GL}_r) = \varinjlim_{\mathcal{U}} H^1(\mathcal{U}/R, \mathrm{GL}_r)$ of non–abelian Čech cohomology of GL_n with respect to the Zariski topology of $\mathrm{Spec}(R)$. By passage to the limit, Lemma **??** implies that $\check{H}^1_{Zar}(R, \mathrm{GL}_r)$ classifies the isomorphism classes of vector bundles of rank r over $\mathrm{Spec}(R)$.

2.4. **Functoriality.** The principle is that nice constructions for vector bundles arise from homomorphisms of group schemes. Given a map $f : \mathrm{GL}_r \to \mathrm{GL}_s$, we can attach to a vector bundle $\mathbf{V}_g$ of rank r (where $g = (g_{i,j})$ is a cocycle) the vector bundle $\mathbf{V}_{f(g)}$ of rank s where $f(g) = (f(g_i))$. This extends to a functor $X \mapsto f_*(X)$ from vector bundles of rank r to vector bundles to rank s. We examine the following three cases.

(a) *Direct sum.* If $r = r_1 + r_2$, we consider the map $f : \mathrm{GL}_{r_1} \times \mathrm{GL}_{r_2} \to \mathrm{GL}_r$, $(A_1, A_2) \mapsto A_1 \oplus A_2$. We then have $f_*(\mathbf{V}_1, \mathbf{V}_2) = \mathbf{V}_1 \oplus \mathbf{V}_2$.

Of course, it can be done with $r = r_1 + \cdots + r_l$, in particular we have in the case $r = 1 + \cdots + 1$ the diagonal map $(\mathbb{G}_m)^r \to \mathrm{GL}_r$ which leads to decomposable vector bundles, that is, direct sum of rank one vector bundles.

(b) *Tensor product.* If $r = r_1 r_2$, we consider the map $f : \mathrm{GL}_{r_1} \times \mathrm{GL}_{r_2} \to \mathrm{GL}_r$, $(A_1, A_2) \mapsto A_1 \otimes A_2$ (called the Knonecker product). We then have $f_*(\mathbf{V}_1, \mathbf{V}_2) = \mathbf{V}_1 \otimes \mathbf{V}_2$.

(c) *Determinant.* We put $\det(\mathbf{V}) = \det_*(\mathbf{V})$, this is the determinant line bundle.

2.5. **The case of a Dedekind ring.** Let R be a Dedekind ring, that is, a noetherian domain such that the localization at each maximal ideal is a discrete valuation ring. The next result is a classical fact of commutative algebra, see [**?**, II.4, thm. 13].

Theorem 2.8. *A locally free R–module of rank $r \geq 1$ is isomorphic to $R^{r-1} \oplus I$ for I an invertible R–module which is unique up to isomorphism.*

Since I is the determinant of $R^{r-1} \oplus I$, the last assertion is clear. Our goal is to discuss this statement with cohomological methods in view of possible generalizations. The key input is the strong approximation theorem for the Dedekind ring R.

Let R_f be localization of R and denote by $\{\mathfrak{p}_1, \dots, \mathfrak{p}_c\} = \operatorname{Spec}(R) \setminus \operatorname{Spec}(R_f)$ and by v_i the discrete valuation of K attached to $\mathfrak{p}_i$. We denote by $\widehat{K}_i$ the completion of K with respect to v_i and bt $\widehat{R}_i$ its valuation ring.

Theorem 2.9. *(1) (Weak Approximation) The image of the diagonal embedding* $K \hookrightarrow \prod_{i=1,\dots,c} \widehat{K}_i$ *is dense.*

(2) (Chinese remainder) For each uple $(e_1, \dots, e_n)$ *of positive integers, the map* $R \to \prod\limits_{i=1}^{n} R/\mathfrak{p}_i^{e_i}$ *is onto and its kernel is* $\prod\limits_{i=1}^{n} \mathfrak{p}_i^{e_i}$.

(3) (Strong approximation) Let $x_1, \dots, x_n \in K$ *and let* $e = (e_1, \dots, e_n)$ *be an uple of integers. Then there exists* $x \in K$ *such that* $v_i(x - x_i) \geq e_i$ *and* $v_{\mathfrak{p}}(x) \geq 0$ *for each maximal ideal* $\mathfrak{p}$ *of* R *satisfying* $\mathfrak{p} \neq \mathfrak{p}_i$ *for* $i = 1, \dots, n$.

Proof. Part (3) implies clearly (1) and (2). For a proof of (3), see [**?**, §I.3] or [**?**, §VII.2.4]. For a direct proof of (2), see [**?**, Tag 00DT]. For a proof of (1), see [**?**, §VI.7.2]. □

Coming back to Theorem **??**, it states firstly that vector bundles over R are decomposable and secondly that vector bundles over R are classified by their determinant. We limit ourself to prove the following corollary by using strong approximation.

Corollary 2.10. *A locally free* R*–module of rank* $r \geq 1$ *is trivial if and only if its determinant is trivial.*

Proof. We are given a vector bundle $\mathbf{V}(M)$. It trivializes over an open affine subset $\operatorname{Spec}(R_f)$ and we put $\Sigma = \operatorname{Spec}(R) \setminus \operatorname{Spec}(R_f) = \{\mathbf{p}_1, \dots, \mathbf{p}_c\}$. We use then the above notations. According to Nakayama's lemma[1], the $\widehat{R}_{\mathbf{p}_i}$–module $M \otimes_R \widehat{R}_{\mathbf{p}_i}$ is free so we can pick a trivialization $\widehat{\phi}_i : (\widehat{R}_{\mathbf{p}_i})^r \xrightarrow{\sim} M \times_R \widehat{R}_{\mathbf{p}_i}$; we bear in mind that the choice $\widehat{\phi}_i$ is up to precompose with an element of $\operatorname{GL}_r(\widehat{R}_{\mathbf{p}_i})$.

On the other hand, let $\phi_f : (R_f)^r \xrightarrow{\sim} M \times_R R_f$ be a trivialization, similarly its choice is up to precompose with an element of $\operatorname{GL}_r(R_f)$. The linear map $\phi_f^{-1} \widehat{\phi}_i$: $(\widehat{K}_{\mathbf{p}_i})^r \to (\widehat{K}_{\mathbf{p}_i})^r$ gives rise to an element $g_i \in \operatorname{GL}_r(\widehat{K}_{\mathbf{p}_i})$. Taking into account the choices, we attached to M an element of the double coset

$$c_\Sigma(R, \operatorname{GL}_r) := \operatorname{GL}_r(R_f) \backslash \prod_{j=1,\dots,c} \operatorname{GL}_r(\widehat{K}_{\mathbf{p}_i}) / \operatorname{GL}_r(\widehat{R}_{\mathbf{p}_i}).$$

Claim 2.11. *The map*

$$\ker\Big(H^1_{Zar}(R, \operatorname{GL}_r) \to H^1_{Zar}(R_f, \operatorname{GL}_r)\Big) \to c_\Sigma(R, \operatorname{GL}_r)$$

[1]We could do it over $R_{\mathbf{p}_i}$ but we want to emphasize that approach involving completions.

is injective.

For the sequel we need only to know that it has trivial kernel. We consider only this special case and let the reader to deal with the general case. Indeed if (g_i) belongs in the kernel, it means that we can adjust the trivializations in order to get $g_i = 1$ for $i = 1, \dots, c$. We claim that the isomorphism $\phi_f : M_f \xrightarrow{\sim} (R_f)^r$ extends (uniquely) to an isomorphism $M \xrightarrow{\sim} R^r$. Since the map $\phi_f : M_f \xrightarrow{\sim} (R_f)^r$ extended over $\widehat{K}_{\mathbf{p}_i}$ is extended of $\widehat{\phi}_i$ by base change of $\widehat{R}_{\mathbf{p}_i}$ to $\widehat{K}_{\mathbf{p}_i}$ it means that there are no denominators involved so that the map extends ϕ_f to an R–linear mapping $\psi : M^r \to R^r$. For the same reason $(\phi_f)^{-1}$ extends as well and we conclude that ϕ_f extends to an R–linear isomorphism $\psi : M^r \xrightarrow{\sim} R^r$.

We assume now that the determinant of $\mathbf{V}(M)$ is trivial so that (g_i) belongs by functoriality to the kernel of the map $\det_* : c_\Sigma(R, \mathrm{GL}_r) \to c_\Sigma(R, \mathbb{G}_m) = R_f^\times \backslash \prod_{j=1,\dots,c} (\widehat{K}_{\mathbf{p}_i}^\times / \widehat{R}_{\mathbf{p}_i}^\times)$. After changing the trivializations we can then assume that $g_i \in \mathrm{SL}_r(\widehat{K}_{\mathbf{p}_i})$ for $i = 1, \dots, c$. Since $\mathrm{SL}_r(\widehat{K}_{\mathbf{p}_i})$ is generated by elementary matrices and since R_f is dense in $\prod_i \widehat{K}_{\mathbf{p}_i}$ by the strong approximation therorem **??**, it follows that $\mathrm{SL}_r(R_f)$ is dense in $\prod_{i=1,\dots,c} \mathrm{SL}_r(\widehat{K}_{\mathbf{p}_i})$ (this goes by decomposing elements in compatible products of elementary matrices). On the other hand, each group $\mathrm{SL}_r(\widehat{R}_{\mathbf{p}_i})$ is open (actually clopen) in $\mathrm{SL}_r(\widehat{K}_{\mathbf{p}_i})$ so that $c_\Sigma(R, \mathrm{SL}_r) = 1$. The Claim **??** enables us to conclude that $\mathbf{V}(M)$ is a trivial vector bundle. □

Remarks 2.12. (a) The general case is quite close; we need to apply the previous argument to $\mathrm{GL}(R^{r-1} \oplus I)$ for an invertible R–module I.

(b) $c_\Sigma(R, \mathbb{G}_m) = \mathrm{Div}_\Sigma(R)/R_f^\times$ is isomorphic to $\ker\bigl(\mathrm{Pic}(R) \to \mathrm{Pic}(R_f)\bigr)$. This is a general fact, i.e. the map of Claim **??** is surjective. This can be seen by using patching techniques.

(c) The density of $\mathrm{SL}_r(R_f)$ in $\prod_{i=1,\dots,c} \mathrm{SL}_r(\widehat{K}_{\mathbf{p}_i})$ is an example of strong approximation. This argument comes from Harder [**?**, Korollar 2.3.2] and is used further (see **??**).

3. Zariski topology is not fine enough

The above definition of non–abelian cohomology extends to an arbitrary group scheme. There are several complementary reasons for trying to extend this theory.

3.1. **The example of quadratic bundles.** A quadratic form on an R–module M is a map $q : M \to R$ which satisfies

(i) $q(\lambda x) = \lambda^2 q(x)$ for all $\lambda \in R$, $x \in M$.

(ii) The form $M \times M \to R$, $(x, y) \mapsto b_q(x, y) = q(x+y) - q(x) - q(y)$ is (symmetric) bilinear.

This concept is stable under arbitrary base change. The form q is *regular* if b_q induces an isomorphism $M \xrightarrow{\sim} M^\vee$. A fundamental example is the hyperbolic form $(V \oplus V^\vee, hyp)$ attached to a locally free R–module of finite rank defined by $hyp(v,\phi) \to \phi(v)$.

Suppose we are given a regular quadratic form (M,q) where M is locally free of rank r. It is tempting to make analogies with vector bundles and to use the orthogonal group scheme $O(q,M)$ which a closed subgroup scheme of $\mathrm{GL}(M)$. More precisely, we have

$$O(q,M)(S) = \big\{ g \in \mathrm{GL}(M)(S), \mid \ q_S \circ g = q_S \big\}$$

for each R–ring S. For an open cover $\mathcal{U}$ of R we define $Z^1(\mathcal{U}/R, O(q,M))$ and $H^1(\mathcal{U}/R, O(q,M))$ in the same way as in section 2 (it makes sense actually for any R–group scheme). What we get is the following.

Lemma 3.1. *The set $H^1_{Zar}(\mathcal{U}/R, O(q,M))$ classifies the isometry classes of regular quadratic forms (q',M') which are locally isomorphic over $\mathcal{U}$ to (q,M).*

Proof. Let $\mathcal{U} = (U_i)_{i \in I}$ be the open cover. We define a class map from the set $\mathcal{S}$ of isomorphism classes of regular quadratic forms (q',M') which are locally isomorphic over $\mathcal{U}$ to (q,M). Let (q',M') be a regular quadratic form such that $(q',M')_{U_i}$ is isometric to $(q,M)_{U_i}$ for each i. In other words we have trivializations map $\phi_i : (q,M)_{U_i} \xrightarrow{\sim} (q',M')_{U_i}$ for each i. On $U_{i,j} = U_i \cap U_j$, we have $g_{i,j} = \phi_i^{-1}\phi_j \in O(q,M)(U_{i,j})$. This is a 1-cocycle, i.e. $g_{i,j} = g_{i,j} g_{j,k}$ on $U_{i,j,k} = U_i \cap U_j \cap U_k$. By taking into account the choices, we obtain a well-defined map $\mathcal{S} \to H^1_{Zar}(\mathcal{U}/R, O(q,M))$. □

This is nice, but the point is that regular quadratic forms over R of dimension r have no reason to be locally isomorphic to (M,q) (e.g. this occurs already with $R = \mathbb{R}$, the field of real numbers). So the set $H^1_{Zar}(R, O(q,M))$ is only a piece of what we would like to obtain.

Remark 3.2. The above dictionnary is an example of the so-called "yoga of forms" which is of generalize nature. See [**?**, §III.2.5] and [**?**, §2.2.4] and §**??**.(d).

3.2. **Functoriality.** If we have a map $f : G \to H$ of group schemes, we would like to have some control on the map $f_* : H^1_{Zar}(R,G) \to H^1_{Zar}(R,H)$.

A basic example is the Kummer map $f_d : \mathbb{G}_m \to \mathbb{G}_m$, $t \mapsto t^d$ for an integer d. It gives rise to the multiplication by d map on the Picard group $\mathrm{Pic}(R)$. In terms of invertible modules, it corresponds to the map $M \mapsto M^{\otimes d}$.

We would like to understand its kernel and its image. We can already say something about the kernel. Given $[M] \in \ker\big(\mathrm{Pic}(R) \xrightarrow{\times d} \mathrm{Pic}(R)\big)$, then there exists a trivialization $\theta : R \xrightarrow{\sim} M^{\otimes d}$. We then define the commutative group $A_d(R)$ of isomorphism classes of couples (M,θ) where M is an invertible R–module equipped with a trivialization $\theta : R \xrightarrow{\sim} M^{\otimes d}$. The multiplication rule is given by $(M,\theta)(M',\theta) = (M \otimes_R M', \widetilde{\theta}$ where $\widetilde{\theta}$ is defined by the composite

$$R \xrightarrow{\sim} R^{\otimes 2} \xrightarrow{\theta \otimes \theta'} M^{\otimes d} \otimes_R M'^{\otimes d} = (M \otimes_R M')^{\otimes d}.$$

The trivial element is (R, θ_0) where $\theta_0 : R \xrightarrow{\sim} R^{\otimes d}$. We have a forgetful map $A_d(R) \to \mathrm{Pic}(R)$.

Lemma 3.2.1. *We have* $d\, A_d(R) = 0$ *and an exact sequence*

$$1 \to R^\times/(R^\times)^d \xrightarrow{\phi} A_d(R) \to \mathrm{Pic}(R) \xrightarrow{\times d} \mathrm{Pic}(R)$$

with $\phi(a) = [(R, \theta_a)]$ *where* $\theta_a : R \xrightarrow{\sim} R^{\otimes d} = R$, $x \mapsto ax$.

Proof. Given $[(M,\theta)] \in A_d(R)$, its d–power is $[(M^{\otimes d}, \theta_d)]$ where $\theta_d : R \xrightarrow{\sim} R^{\otimes d} \xrightarrow{\theta^{\otimes d}} (M^{\otimes d})^{\otimes d} = M^{\otimes d^2}$. It follows that $(M^{\otimes d}, \theta_d)$ is isomorphic to (R, θ_0).

Next assume that $\phi(a) = [(R, \theta_a)]) = 0 \in A_d(R)$, that is, there exists an isomorphism $\phi : R \xrightarrow{\sim} R$ of R–modules such that $\phi_* \theta_0 = \theta_a$. The map ϕ is the multiplication by $b \in R^\times$ so that $b^d = a$. The injectivity of the first map is established.

Clearly the sequence $R^\times/(R^\times)^d \xrightarrow{\phi} A_d(R) \to \mathrm{Pic}(R)$ is a complex, let us prove its exactness. We are given (M, θ) such that $R \cong M$ so that we can deal with (R, θ). Then $\theta : R \xrightarrow{\sim} R^{\otimes d} = R$ is given by $a \in R^\times$. Therefore $(R,\theta) = (R, \theta_a)$. Finally the exactness at $\mathrm{Pic}(R)$ is obvious. □

We will see later that we can provide a cohomological meaning to the group $A_d(R)$ (Remark **??**).

4. General definitions

Grothendieck-Serre's idea is to extend the notion of covers in algebraic geometry [**?**]. They did it originally with étale covers (discussed in §**??**) but it turns out that the flat cover setting is simpler in a first approach. This is the setting of the book by Demazure-Gabriel [**?**, §III], and there are variants.

4.1. Non-abelian Čech cohomology.

Definition 4.1. *A flat (or fppf= fidèlement plat de présentation finie) cover of* R *is a finite collection* $(S_i)_{i \in I}$ *of* R*–rings satisfying*

(i) S_i *is a flat* R*–algebra of finite presentation for* $i = 1, \dots, c$;

(ii) $\mathrm{Spec}(R) = \bigcup_{i \in I} \mathrm{Im}\Big(\mathrm{Spec}(S_i) \to \mathrm{Spec}(R)\Big)$.

If we put $S = \prod_{i \in I} S_i$, the conditions rephrase by saying that S is a faithfully flat R–algebra of finite presentation. We can therefore always deal with a unique ring.

Remark 4.2. *For a partition* $1 = f_1 + \dots + f_n$, *the family* $(R_{f_j})_{j=1,\dots,n}$ *is a flat cover of* R *and so is* $R_{f_1} \times \dots \times R_{f_n}$.

We define now non-abelian cohomology. Let S be a faithfully flat R–algebra of finite presentation. We denote by $p_i^* : S \to S \otimes_R S$ the coprojections $(i = 1, 2)$ and similarly $q_i^* : S \to S \otimes_R S \otimes_R S$ $(i = 1, 2, 3)$, $q_{i,j}^* : S \otimes_R S \to S \otimes_R S \otimes_R S$ the partial coprojections $(i < j)$.

Let G be an R–group scheme. A 1-*cocycle* for G and S/R is an element $g \in G\bigl(S \otimes_R S\bigr)$ satisfying

$$q_{1,2}^*(g)\, q_{2,3}^*(g) = q_{1,3}^*(g) \quad \in \quad G\bigl(S \otimes_R S \otimes_R S\bigr).$$

We denote by $\mathrm{Z}^1(S/R, G)$ the pointed set of 1-cocycles of S/R with values in G (it is pointed by the trivial 1–cocycle).

Two such cocycles $g, g' \in G(S)$ are *cohomologous* if there exists $h \in G(S)$ such that $g = p_1^*(h^{-1})\, g'\, p_2^*(h)$. We denote by $\check{\mathrm{H}}^1(S/R, G) = \mathrm{Z}^1(S/R, G)/\sim$ the pointed set of 1-cocycles up to cohomology equivalence.

Remark 4.3. In the case of a Zariski cover given by a partition of 1, the definition is the same as in §**??**. What lies behind this, is the fact that intersection of open subschemes is a special case of fiber product.

We can pass to the limit on all flat covers of Spec(R) and define $\check{\mathrm{H}}^1_{fppf}(R, G) = \varinjlim \check{\mathrm{H}}^1(S/R, G)$ [2]. This construction is functorial in R and in the group scheme G.

4.2. **Torsors.** A (right) G–torsor X (with respect to the flat topology) is an R-scheme equipped with a right action of G which satisfies the following properties:

(i) the action map $X \times_R G \to X \times_R X$, $(x, g) \mapsto (x, x.g)$, is an isomorphism;

(ii) There exists a flat cover R'/R such that $X(R') \neq \emptyset$.

The first condition reflects the simple transitivity of the action, i.e. $G(T)$ acts simply transitively on $X(T)$ for all R–rings T. The second condition is a local triviality condition. An example is $X = G$ with G acting by right translations, it is called the split G–torsor.

If $X(R) \neq \emptyset$, a point $x \in X(R)$ defines a morphism $G \to X$, $\phi_x : g \mapsto x.g$ which is an isomorphism by the simply transitive property; we say that X is trivial and that ϕ_x is a trivialization.

Condition (ii) rephrases that an R–torsor X under G is locally trivial for the flat topology.

A morphism of G–torsors $X \to Y$ is a G–equivariant map; once again the simple transitivity condition shows that such a morphism is an isomorphism. Thus the category of G–torsors is a groupoid.

The R–functor of automorphisms of the trivial G–torsor G is representable by G (acting by left translations).

[2]There are set-theoretic issues there allowing us to consider this limit, see [**?**, Remarque 1.4.3] and [**?**] for the fpqc setting.

We denote by $H^1_{fppf}(R,G)$ the set of isomorphism classes of G–torsors for the flat topology. If S is a flat cover R, we denote by $H^1_{fppf}(S/R,G)$ the subset of isomorphism classes of G–torsors trivialized over S.

As in the vector bundle case, we shall construct a class map $\gamma : H^1_{fppf}(S/R,G) \to \check{H}^1_{fppf}(S/R,G)$ as follows.

Let X be a G–torsor over R equipped with a trivialization $\phi : G \times_R S \xrightarrow{\sim} X \times_R S$. Over $S \otimes_R S$, we then have two trivializations $p_1^*(\phi) : G \times_R (S \otimes_R S) \xrightarrow{\sim} X \times_R (S \otimes_R S)$ and $p_2^*(\phi)$. It follows that $p_1^*(\phi)^{-1} \circ p_2^*(\phi)$ is an automorphism of the trivial G–torsor over $S \otimes_R S$, so is the left translation by an element $g \in G(S \otimes_R S)$. A computation shows that g is a 1–cocycle [**?**, §2.2]; also changing ϕ changes g by a cohomologous cocycle. The class map is then well-defined. Its study involves a glueing technique in the flat setting.

4.3. **Interlude: Faithfully flat descent.** Let T be a faithfully flat extension of the ring R (not necessarily of finite presentation). We put $T^{\otimes d} = T \otimes_R T \cdots \otimes_R T$ (d times). One first important thing is that the Amitsur complex

$$0 \to M \to M \otimes_R T \xrightarrow{d_2} M \otimes_R T \otimes_R T \xrightarrow{d_2} M \otimes_R T^{\otimes 3} \dots$$

is exact for each R–module M [**?**, III.1] where

$$d_n(m \otimes t_1 \otimes \cdots \otimes t_n) = \sum_{i=0,\dots,n} (-1)^i m \otimes t_1 \otimes \cdots \otimes t_i \otimes 1 \otimes t_{i+1} \otimes \cdots \otimes t_n.$$

This implies in particular that for any affine R-scheme X, we have an identification

$$X(R) = \big\{ x \in X(T) \mid p_1^*(x) = p_2^*(x) \in X(T \otimes_R T) \big\}$$

which holds actually for any R-scheme. Given a T–module N we consider the $T \otimes_R T$–modules $p_1^*(N) = T \otimes_R M$ and $p_2^*(N) = M \otimes_R T$.

A descent datum on N is an isomorphism $\varphi : p_1^*(N) \xrightarrow{\sim} p_2^*(N)$ of $T^{\otimes 2}$–modules such that the diagram

$$\begin{array}{ccc} T \otimes_R T \otimes_R N & \xrightarrow{\varphi_3} & N \otimes_R T \otimes_R T \\ & \searrow^{\varphi_2} \quad\quad \nearrow_{\varphi_1} & \\ & T \otimes_R N \otimes_R T & \end{array}$$

is commutative where

- $\varphi_3(t_1 \otimes t_2 \otimes n) = \varphi(t_1 \otimes n) \otimes t_2$;
- $\varphi_2(t_1 \otimes t_2 \otimes n) = t_2 \otimes \varphi(t_1 \otimes n)$;
- $\varphi_1(t_1 \otimes n \otimes t_3) = t_1 \otimes \varphi(n \otimes t_3)$

There is an obvious notion of morphisms for T–modules equipped with a descent datum from T to R. If M is an R–module, the identity of M gives rises to a canonical isomorphism $can_M : p_1^*(M \otimes_R T) \xrightarrow{\sim} p_2^*(M \otimes_R T)$, this is a descent datum.

Theorem 4.4. *(Faithfully flat descent, see* [**?**, III, th. 2.1.2] *)*
(1) The functor $M \to (M \otimes_R T, can_M)$ *is an equivalence of categories between the category of* R*–modules and that of* T*–modules with descent datum. An inverse functor (the descent functor) is* $(N, \varphi) \mapsto \{n \in N \mid n \otimes 1 = \varphi(1 \otimes n)\}$.
(2) The functor above induces an equivalence of categories between the category of R*–algebras (commutative, unital) and that of* T*–algebras (commutative, unital) with descent datum.*

For an exhaustive view, we recommend [**?**, Tag 023F]. We shall later see examples of descent beyond the case of Zariski covers (e.g. **??**).

4.4. **The linear case.** An important example is the extension of Swan-Serre's correspondence. A consequence of the faithfully flat descent theorem (and of the fact that the property to be locally free of rank r is local for the flat topology [**?**, Tag 05B2], [**?**, III.2.8]) is the following.

Theorem 4.5. *Let* $r \geq 0$ *be an integer.*
(1) Let M *be a locally free* R*–module of rank* r*. Then the* R*–functor* $S \mapsto \mathrm{Isom}_{S-mod}(S^r, M \otimes_R S)$ *is representable by a* GL_r*–torsor* X^M *over* $\mathrm{Spec}(R)$.
(2) The functor $M \mapsto X^M$ *induces an equivalence of categories between the groupoid of locally free* R*–modules of rank* r *and the category of* GL_r*–torsors over* $\mathrm{Spec}(R)$.

Proof. See [**?**, 2.4.3.1]. This reference is for Zariski topology and étale topology but works for the flat topology in view of the postponed Proposition **??**. □

This implies that the GL_r–torsors are the same with flat topology or with Zariski topology.

Corollary 4.6. *(Hilbert-Grothendieck 90) We have* $H^1_{Zar}(R, \mathrm{GL}_r) = H^1_{fppf}(R, \mathrm{GL}_r)$*. In particular, if* R *is a local (or semilocal) ring, we have* $H^1_{fppf}(R, \mathrm{GL}_r) = 1$.

This is a special case of a more general statement which holds for $\mathrm{GL}_1(\mathcal{B})$ where $\mathcal{B}$ is an Azumaya R–algebra see [**?**, §4.2]. More generally, it holds for a separable R–algebra (for example Azumaya or finite étale) which is a locally free R–module of finite rank.

4.5. **Torsors and cocycles.**

Lemma 4.7. *The map* $\gamma : H^1_{fppf}(S/R, G) \to \check{H}^1_{fppf}(S/R, G)$ *is injective.*

Proof. Once again we limit ourselves to the kernel for simplicity (for the general argument, see [**?**, §2.2]). If (X, ϕ) gives rise to a cocycle which is cohomologous to the trivial cocycle, it means that there exists a trivialization $\phi' : G \times_R S \xrightarrow{\sim} X \times_R S$ such that the associated cocycle is trivial. We put $x = \phi'(1) \in X(S)$. Then $p_1^*(x) = p_2^*(x) = 1$. Since $X(R)$ identifies with $\{x \in X(S) \mid p_1^*(x) = p_2^*(x)\}$, we conclude that $X(R)$ is non-empty. □

Theorem 4.8. *If G is affine, the class map $H^1_{fppf}(S/R, G) \to \check{H}^1_{fppf}(S/R, G)$ is an isomorphism.*

Note that by passing to the limit on the flat covers, we get a bijection $H^1_{fppf}(R, G) \to \check{H}^1_{fppf}(R, G)$. The fact that we can descend torsors under an affine group scheme is a consequence of the faithfully flat descent theorem. The sketch is as follows where we denote by $R[G]$ the coordinate ring of G. We are given a cocycle $g \in G(S \otimes_R S)$. We consider the map $L_g^* : (S \otimes_R S)[G] \xrightarrow{\sim} (S \otimes_R S)[G]$ (where $L_g : G \to G$ is the left multiplication by g) define φ_g by the diagram

$$\begin{array}{ccc} S \otimes_R S[G] & \xrightarrow[\sim]{\varphi_g} & S[G] \otimes_R S \\ \cong \Big\downarrow \alpha & & \cong \Big\downarrow \beta \\ (S \otimes_R S)[G] & \xrightarrow[\sim]{L_g^*} & (S \otimes_R S)[G] \end{array}$$

where $\alpha(s_1 \otimes f) = (s_1 \otimes 1) p_2^*(f)$ and $\beta(f \otimes s_2) = p_1^*(f)(1 \otimes s_2)$. The cocycle condition implies that ϕ_g is a descent datum for the S–algebra $S[G]$. Theorem **??** defines an R–algebra $R[X]$ and X is actually a G–torsor denoted by E_g.

This construction is a special case of *twisting*. More generally, if Y is an affine R–scheme equipped with a left action of G, then the action map $g : Y \times_R (S \otimes_R S) \xrightarrow{\sim} Y \times_R (S \otimes_R S)$ defines a descent datum. This gives rises to the twist of Y_g of Y by the 1-cocycle g. The scheme Y_g is affine over R.

A special case is the action of G on itself by inner automorphisms, G_g is called the twisted R–group scheme; it acts on Y_g for Y as above.

Remarks 4.9. (a) The above construction do not depend of choices of trivializations. We can define for a G–torsor E the twist ${}^E Y$ and ${}^E G$ by means of contracted products.

(b) In practice, the affiness assumption in Theorem **??** is too strong. More generally we can twist G–schemes equipped with an ample invertible G-linearized bundle, see [**?**, §6, 7 and §10, lemma 6] for details.

4.6. **Examples.** *(a) Vector group schemes.* Let M be a locally free R–module of finite rank, we claim that $\check{H}^1(R, \mathbf{W}(M)) = 0$ so that each $\mathbf{W}(M)$–torsor is trivial. We are given a flat cover S/R. Since the complex

$$M \otimes_R S \xrightarrow{p_1^* - p_2^*} M \otimes_R S \otimes_R S \to M \otimes_R S \otimes_R S \otimes_R S$$

is exact, each cocycle $g \in \mathbf{W}(M)(S \otimes_R S) = M \otimes_R S \otimes_R S$ is a coboundary. Thus $\check{H}^1(S/R, \mathbf{W}(M)) = 0$ and $\check{H}^1(R, \mathbf{W}(M)) = 0$.

(b) An important case is $G = \Gamma_R$, that is, the *finite constant group scheme* attached to an abstract finite group Γ. Recall that $G(S)$ is the group of locally constant functions $\mathrm{Spec}(S) \to \Gamma$. In other words, $G = \sqcup_{\gamma \in \Gamma} \mathrm{Spec}(R)_\gamma$ so that its coordinate ring identifies with $R^{(\Gamma)}$.

In this case a Γ_R–torsor $\mathrm{Spec}(S) \to \mathrm{Spec}(R)$ is the same thing as a Galois Γ–algebra S and is called often a Galois cover[3]. A special case is that of a finite Galois extension L/k of fields of Galois group Γ.

(c) As for GL_r, a special nice case is the case of *forms*, that is when G is the automorphism group of some algebraic structure, see [**?**, §2.2.3] for an exhaustive discussion.

For example, the orthogonal group scheme O_{2n} is the automorphism group of the hyperbolic quadratic form attached to R^n. As regular quadratic forms of rank $2n$ are locally isomorphic to the hyperbolic form in the flat topology, descent theory provides an equivalence of categories between the groupoid of regular quadratic forms of rank $2n$ and $H^1_{fppf}(R, O_{2n})$. This is what we wanted in §**??**, that is, $H^1(R, O_{2n})$ classifies the isomorphism classes of regular quadratic R–forms of rank $2n$ [**?**, III.5.2].

(d) Another important example is that of the symmetric group S_n. For any R–algebra S, the group $S_n(S)$ is the automorphism group of the S–algebra $S^n = S \times \cdots \times S$ (n–times). Since finite étale algebras of degree n are locally isomorphic to R^n for the étale topology, the same yoga shows that there is an equivalence of categories between the category of S_n–torsors and that of finite étale R–algebras of rank n.

The functor which associates to a finite étale R–algebra of rank n a S_n–torsor is defined by descent but can be described explicitely. This is the Galois closure construction done by Serre in [**?**, §1.5], see also [**?**].

4.7. **Functoriality issues.** Let $G \to H$ be a monomorphism of R–group schemes. We say that an R-scheme X equipped with a map $f : H \to X$ is a *flat quotient* of H by G if for each R–algebra S the map $H(S) \to X(S)$ induces an injective map $H(S)/G(S) \hookrightarrow X(S)$ and if for each $x \in X(S)$, there exists a flat cover S' of S such that $x_{S'}$ belongs to the image of $H(S') \to X(S')$ (we say that f is "couvrant" in French). If it exists, a flat quotient is unique (up to unique isomorphism); furthermore, if G is normal in H, then X carries a natural structure of R–group schemes, we say in this case that $1 \to G \to H \to X \to 1$ is an exact sequence of R–group schemes (for the flat topology).

Lemma 4.10. *Assume that X is the flat quotient of H by G.*

(1) The map $H \to X$ is a G–torsor.

(2) There is an exact sequence of pointed sets

$$1 \to G(R) \to H(R) \to X(R) \xrightarrow{\varphi} H^1_{fppf}(R, G) \to H^1_{fppf}(R, H)$$

where $\varphi(x) = [f^{-1}(x)]$.

For the proof, see [**?**, III.4.2, cor. 1.8 and III.4.4].

[3]This is our convention for Galois covers which has the advantage to be stable for base change. In [**?**], one requires furthermore R, S to be connected.

Remark 4.11. (a) Assume that X is affine (or is equipped with an ample G–linearized invertible sheaf, see [**?**, §6.1, thm. 7 and §10, lemma 6] for details). Then the category of G–torsors over $\mathrm{Spec}(R)$ is equivalent to the category of couples (F, x) where F is a H–torsor and $x \in ({}^F X)(R)$ (where ${}^F X$ is the twist of X by the H–torsor F).

(b) If G is normal in H, then X has natural structure of R–group scheme. In this case (a) rephrases by saying that the category of G–torsors over $\mathrm{Spec}(R)$ is equivalent to the category of couples (F, ϕ) where F is a H–torsor and ϕ a trivialization of the X–torsor ${}^F X$.

(c) Using the extended Swan-Serre correspondence **??**, an example of (b) is that category of SL_r-torsors is equivalent to the category of pairs (M, θ) where M is a locally free R–module of rank r and $\theta : R \xrightarrow{\sim} \Lambda^r(M)$ is a trivialization of the determinant of M.

(d) For an integer d, we have the Kummer exact sequence $1 \to \mu_d \to \mathbb{G}_m \xrightarrow{\times d} \mathbb{G}_m \to 1$. Similarly the category of μ_d–torsors is equivalent to the category of pairs (M, θ) where M is an invertible R–module and $\theta : R \xrightarrow{\sim} M^{\otimes r}$ a trivialization. This is related with §**??**.

Examples 4.7.1. $\mathbb{G}_m$ is the flat quotient of GL_r by SL_r and $\mathbb{G}_m$ is the flat quotient of $\mathbb{G}_m$ by μ_d.

There are of course many more functorial properties for example when G is commutative. In this case, $H^1(R, G)$ is equipped with a natural structure of abelian group arising from the product morphism $G \times_R G \to G$.

4.8. **Étale covers.** We remind to the reader that an étale morphism of rings $R \to S$ is a smooth morphism of relative dimension zero [**?**, §I.3]. There are several alternative definitions, for example, S is a flat R–algebra of finite presentation such that for each R–field F, then $S \otimes_R F$ is an étale F–algebra (i.e. a finite geometrically reduced F–algebra).

Examples 4.12. *(a) A localization morphism $R \to R_f$ is étale.*

(b) If d is invertible in R, the Kummer morphism $\mathbb{G}_m \to \mathbb{G}_m$, $t \mapsto t^d$ is étale.

(c) More generally, if d is invertible in R and $r \in R^\times$, then $S = R[x]/(x^d - r)$ is a finite étale R–algebra.

For an R–group scheme G, we define the subset $H^1_{ét}(R, G)$ of $\check{H}^1_{fppf}(R, G)$ of classes ot torsors which are trivialized by an étale cover. We define similarly $\check{H}^1_{ét}(R, G)$

Proposition 4.13. *If G is affine smooth, then we have $H^1_{ét}(R, G) = H^1_{fppf}(R, G)$.*

Proof. Smoothness is a local property with respect to the flat topology so that any G–torsor E is smooth affine over R. According to the existence of quasi-sections [**?**, 17.16.3], E admits locally sections with respect to the étale topology. □

4.9. **Isotrivial torsors and Galois cohomology.** We are given a Galois R–algebra S of group Γ. The action isomorphism $\mathrm{Spec}(S) \times_R \Gamma_S \xrightarrow{\sim} \mathrm{Spec}(S) \times_R \mathrm{Spec}(S)$ can be viewed as an isomorphism $S \otimes_R S \xrightarrow{\sim} S \otimes_R R^{(\Gamma)} = S^{(\Gamma)}$. A 1-cocycle is then an element $z = (z_\gamma)_{\gamma \in \Gamma} \in G(S \otimes_R S) = G(S)^{(\Gamma)}$ satisfying a certain relation.

Since Γ acts on the left on S, it acts as well on the left on $G(S)$.

Lemma 4.14. *(see [?, lemme 2.2.3]) A Γ-uple $z = (z_\sigma)_{\sigma \in \Gamma} \in G(S^{(\Gamma)}) = G(S)^{(\Gamma)} = \mathrm{Hom}_{sets}(\Gamma, G(S))$ is a 1–cocycle for S/R if and only if*

$$z_{\sigma\tau} = z_\sigma \, \sigma(z_\tau)$$

for all $\sigma, \tau \in \Gamma$.

We find that $Z^1(S/R, G)$ is the set of Galois cocycles $Z^1(\Gamma, G(S))$ and that $\check{H}^1(S/R, G)$ is the set of non-abelian Galois cohomology $H^1(\Gamma, G(S)) = Z^1(\Gamma, G(S))/\sim$ where two cocycles z, z' are cohomologous if $z_\gamma = g^{-1} \, z'_\gamma \, \gamma(g)$ for some $g \in G(S)$.

An interesting case is that of a constant group scheme G associated to an abstract group Θ and S is connected. In this case, we have $Z^1(S/R, G) = \mathrm{Hom}_{gp}(\Gamma, \Theta)$ and $\check{H}^1(S/R, G) = \mathrm{Hom}_{gp}(\Gamma, \Theta)/\Theta$.

Remark 4.15. Galois descent is therefore a special case of faithfully flat descent. The reader can check that the category of R-modules is equivalent to the category of couples (N, ρ) where N is a S-module equipped with a semilinear action of Γ (i.e. $\rho(\sigma)(\lambda \, . \, n) = \sigma(\lambda) \, . \, \rho(\sigma)(n)$).

We say that torsor E under an R–group scheme G is isotrivial if it is split by a finite étale cover (which can be assumed Galois up to take the Galois closure). This is subclass of torsors which can be explicitly studied by Galois cohomology computations. This is often a preliminary question to decide whether a given torsor is isotrivial. For example, for the ring of Laurent polynomials in characteristic zero and a reductive group scheme, this is the case [?].

5. Torsors over affine curves

5.1. **The Dedekind case.** Let R be a Dedekind ring with fraction field K. Let $f \in R$ and put $\Sigma = \mathrm{Spec}(R) \setminus \mathrm{Spec}(R_f) = \{\mathbf{p}_1, \dots, \mathbf{p}_c\}$, and use the notation of the proof Corollary **??**. Let G be an affine flat R–group scheme. As in the proof of **??** we have a class map

$$\ker\Big(H^1_{fppf}(R, G) \to H^1_{fppf}(R_f, G) \times \prod_{i=1,\dots,c} H^1_{fppf}(\widehat{R}_{\mathbf{p}_i}, G) \Big) \\ \to c_\Sigma(R, G) = G(R_f) \backslash \prod_{j=1,\dots,c} G(\widehat{K}_{\mathbf{p}_i}) / G(\widehat{R}_{\mathbf{p}_i}).$$

This map is injective [?, §2.3]. The next results are due to Harder [?, Cor. 2.3.2 and Satz 3.3].

Corollary 5.1. *If $c_\Sigma(R,G)=1$ (in particular if $G(R_f)$ is dense in $\prod_{j=1,\dots,c} G(\widehat{K}_{\mathbf{p}_i})/G(\widehat{R}_{\mathbf{p}_i})$), we have* $\ker\Big(H^1_{fppf}(R,G)\to H^1_{fppf}(R_f,G)\times \prod_{i=1,\dots,c} H^1_{fppf}(\widehat{R}_{\mathbf{p}_i},G)\Big)=1$.

Proposition 5.2. *Assume that G is a semisimple split R–group scheme and let (B,T) be a Killing couple, i.e. T is a maximal split R–torus of G and B a R–Borel subgroup scheme containing it.*

(1) The sequence of pointed sets

$$H^1_{fppf}(R,T)\to H^1_{fppf}(R,G)\to H^1_{fppf}(F,G)$$

is exact.

(2) If G is simply connected, then $\ker\Big(H^1_{fppf}(R,G)\to H^1_{fppf}(F,G)\Big)=1$.

At this stage we need to explain the vocabulary for semisimple algebraic groups and also for group schemes. A reference is [**?**, §1.5 and Exercise 6.5.2].

• An affine smooth connected affine algebraic group G defined over an algebraically closed field k is semisimple if 1 is the only smooth connected k–subgroup which is normal and solvable. Simply connected here is more complicated; in characteristic zero this is equivalent to say that G is simply connected for Grothendieck's theory [**?**] of finite étale covers [4]. Examples of semisimple simply connected algebraic groups are SL_n, Sp_{2n}, Spin_n.

• A smooth affine group scheme G over a ring R is *semisimple* (resp. semisimple simply connected) if each geometric fiber $G_{\overline{s}}$ is semisimple (resp. semisimple simply connected).

• [**?**, 5.1.1] Let G be a semisimple group scheme over a connected ring R. It is *split* if there exists a maximal torus $T\cong \mathbb{G}_m^r$ such that each root space $\mathrm{Lie}(G)_a$ for $a\in\widehat{T}$ is free of rank 1 over R, It admits a Borel R–subgroup scheme (i.e. a closed smooth R–subgroup whose geometric fibers are Borel subgroups) containing T [**?**, XXII.5.1.1].

Proof. (1) Since $H^1_{fppf}(K,T)=1$ (Hilbert 90), the sequence $H^1_{fppf}(R,T)\to H^1_{fppf}(R,G)\to H^1_{fppf}(F,G)$ is a complex of pointed sets. In order to establish the exactness, we claim first that the map

$$H^1(R,B)\to \ker\big(H^1_{fppf}(R,G)\to H^1_{fppf}(K,G)\big)$$

is onto. Let E be a R–torsor under G which becomes trivial over K. We admit that the fppf sheaf G/B is representable by a projective R–group scheme [**?**, XXVI.1.2]. The idea is to introduce the twisted R-scheme $Y=E(G/B)$ (it is the scheme of Borel subgroups of the twisted R–group scheme $E(G)$ so is projective over R [**?**, XXVI.3]). Since E_K is trivial we have $Y(K)\neq\emptyset$. Next we have $Y(R)=Y(K)$ in view of the valuative criterion of properness. It follows that Y has an R–point (equivalently $E(G)$

[4]In particular, for the field of complex numbers, this notion coincide with the topological one

carries a R-Borel subgroup scheme). According to Remark **??**.(a), it follows that $[E]$ belongs to the image of $H^1(R,B) \to H^1(R,G)$.

We have $B = U \rtimes T$ where U admits a T–equivariant filtration $U_0 = 1 \subset \cdots \subset U_r = U$ such that U_{i+1}/U_i is isomorphic to $\mathbb{G}_a^{l_i}$. Since $H^1_{fppf}(R,\mathbb{G}_a) = 1$ (Example **??**.(a)), a dévissage argument shows that the map $H^1_{fppf}(R,T) \to H^1_{fppf}(R,B)$ is bijective[5]. We conclude that $[E]$ belongs to the image of $H^1_{fppf}(R,T) \to H^1_{fppf}(R,G)$.

(2) We assume now that G is semisimple simply connected. Taking an isomorphism $T \cong \mathbb{G}_m^r$, we have $H^1_{fppf}(R,T) \cong \mathrm{Pic}(R)^r$. In view of (1), $\ker\big(H^1_{fppf}(R,G) \subset H^1_{fppf}(K,G)\big) \subset H^1_{Zar}(R,G)$. In particular for $[E] \in \ker\big(H^1_{fppf}(R,G) \subset H^1_{fppf}(K,G)\big)$, there exists $f \in R$ such that E_{R_f} is trivial as well with $E_{R_{\mathfrak{p}_i}}$ for the maximal ideals $\mathfrak{p}_1, \dots, \mathfrak{p}_c \in \Sigma \,\mathrm{Spec}(R) \setminus \mathrm{Spec}(R_f)$. It makes then sense to consider the class map of $[E]$ in $c_\Sigma(R,G)$.

Claim 5.1.1. $c_\Sigma(R,G) = 1$.

The Claim and Corollary **??** implies that $[E] = 1 \in H^1_{fppf}(R,G)$ as desired. To establish the Claim, we consider an opposite Borel R–subgroup B^- to B, i.e. $T = B \cap B^-$ [**?**, prop. 5.2.12]. We denote by U^- its unipotent radical. Since each $G(\widehat{K}_{p_i})$ is generated by $U^+(\widehat{K}_i)$ and $U^-(\widehat{K}_i)$ [**?**, lemma 64] and since U^+ (resp. U_-) is isomorphic as R–scheme to $\mathbb{A}^n$, we have that $U^+(R_f)$ is dense in $\prod_{i=1} U^+(\widehat{K}_{\mathfrak{p}_i})$ and similarly for U^-. It follows that $G(R_f)$ is dense in $\prod_{i=1} U^+(\widehat{K}_{\mathfrak{p}_i})$ whence the Claim. □

We find then one more time than $H^1_{Zar}(R,\mathrm{SL}_n) = 1$ but get for example that $H^1_{Zar}(R,E_8) = 1$ where E_8 stands for the split group of type E_8. Since $\mathrm{Pic}(k[t]) = 0$ for a field k, it follows that $H^1_{Zar}(k[t],G) = 1$ for a semisimple k–group G.

5.2. **Affine curves over an algebraically closed field.**

Theorem 5.3. *Let G be a semisimple algebraic k–group where k is an algebraically closed field. Let C be a smooth connected affine curve. Then $H^1_{fppf}(C,G) = 1$.*

A slightly more general version is avalaible in [**?**, §3]. One first ingredient is Steinberg's theorem.

Theorem 5.4. [**?**, Thm. 11.1] *Let F be a field and let H be a semisimple algebraic F–group which is quasi-split (i.e. admits a Borel F–subgroup). Then the map*

$$\bigsqcup_{T \subset H} H^1(F,T) \to H^1(F,H)$$

is onto where T runs over the maximal F–tori of H.

For the field $k(C)$, we have that $\mathrm{Br}(k(C)) = 0$ and more generally that $\mathrm{cd}(k(C)) = 1$, this is a consequence of Tsen's theorem stating that $k(C)$ has the C_1 property [**?**, II.3.3]. A classical dévissage yields that that $H^1(k(C),T) = 1$ for each $k(C)$–torus

[5] this is a general fact, see [**?**, XXVI.2.3].

T[6]. Comining with Theorem **??** yields that $H^1(k(C),G)=1$ for each semisimple (split) k–group G. A special case is that of PGL_n which can be rephrased by saying that the central simple algebras over $k(C)$ are matrix algebras.

A second ingredient is the fact that the Picard group $\mathrm{Pic}(C)$ is divisible which follows from the structure of $\mathrm{Pic}(C^c)$ where C^c is a smooth compactification of C, i.e. an exact sequence $0 \to J_{C^c}(k) \to \mathrm{Pic}(C^c) \to \mathbb{Z} \to 0$ where J_{C^c} is the Jacobian variety of C^c [**?**, REF] (or [**?**, Tag 03RN]). If $C = C^c \setminus \{x_1, \dots, x_s\}$ the surjective map $\mathrm{Pic}(C^c) \to \mathrm{Pic}(C)$ induces an epimorphism $J_{C^c}(k) \twoheadrightarrow \mathrm{Pic}(C)$. Thus $\mathrm{Pic}(C)$ is divisible.

We proceed now to the proof of Theorem **??**.

Proof. We assume first that G is simply connected. Proposition **??** shows that $\ker\bigl(H^1_{fppf}(C,G) \to H^1_{fppf}(k(C),G)\bigr) = 1$. Since $H^1(k(C),G)=1$, it follows that $H^1_{fppf}(C,G)=1$.

For the general case, let $f : G^{sc} \to G$ be the simply connected cover of G (e.g. $\mathrm{SL}_n \to \mathrm{PGL}_n$, $\mathrm{Spin}_n \to \mathrm{SO}_n$) and put $\mu = \ker(f)$. Let T^{sc} be a maximal torus of G^{sc}, then $T = T^{sc}/\mu$ is a maximal torus of G. We consider the commutative diagram

$$\begin{array}{ccc} H^1_{fppf}(C,T^{sc}) & \xrightarrow{\quad f_* \quad} & H^1_{fppf}(C,T) \\ \downarrow & & \downarrow\!\!\!\downarrow \\ 1 = H^1_{fppf}(C,G^{sc}) & \longrightarrow & H^1_{fppf}(C,G). \end{array} \tag{5.1}$$

The surjectivity of the vertical map follows from $H^1(k(C),G)=1$ and Proposition **??**.(1). We use now the exact sequence $1 \to \mu \to T^{sc} \xrightarrow{f} T \to 1$. We choose isomorphisms $T^{sc} \cong \mathbb{G}_m^r$ and $T \cong \mathbb{G}_m^r$, f is given by a map $A : \mathbb{Z}^r \to \mathbb{Z}^r$ (on the cocharacters) such that $\det(A) \in \mathbb{Q}^\times$ It follows that f_* reads

$$A : \mathrm{Pic}(C)^r \to \mathrm{Pic}(C)^r.$$

Since $\det(A) \in \mathbb{Q}^\times$ and $\mathrm{Pic}(C)$ is divisible, the map f_* is then onto. Diagrame chase in the diagram (**??**) enables us to conclude that $H^1(C,G)=1$. □

Remark 5.5. The reductive case is of the same vein. Let $S = G/DG$ be the coradical torus of G. One can show that the map $H^1(C,G) \to H^1(C,S)$ is bijective. This generalizes the bijection $H^1(C,\mathrm{GL}_r) \xrightarrow{\sim} H^1(C,\mathbb{G}_m) = \mathrm{Pic}(C)$ seen in Theorem **??**.

5.3. **The case of the affine line.**

Theorem 5.6. *(Raghunathan-Ramanathan [**?**]) Let G be a reductive k–group over a field k. Then we have a bijection*

$$H^1(k,G) \xrightarrow{\sim} \ker\bigl(H^1(k[t],G) \to H^1(k_s[t],G)\bigr).$$

[6]Hint: Let n be the degree of a splitting field of T, show that $nH^1(k,T)=1$ and consider the exact sequence $1 \to {}_nT \to T \xrightarrow{\times n} T \to 1$.

If k is perfect or if the characteristic of p is "good" for G, we have $H^1(k_s[t],G)=1$ so that $H^1(k,G)=H^1(k[t],G)$. When it happens, we say that G–torsors over $k[t]$ are constant. There are a few exotic cases when it does not hold (for example $G=\mathrm{PGL}_p$ with k imperfect of characteristic $p>0$). The common ingredient of all proofs of this statement is to use Grothendieck-Harder's theorem on bundles over the projective line.

Theorem 5.7. *(see [?, I.1.2.1]) Let G be a reductive k–group over a field k. Let S be a maximal k–split torus of G and let $W_G(S)=N_G(S)/C_G(S)$ be the finite (constant) associated Weyl group. Then we have a bijection*

$$H^1_{Zar}(\mathbb{P}^1_k,S)/W_G(S) \xrightarrow{\sim} \ker\big(H^1(\mathbb{P}^1_k,G) \xrightarrow{ev_0} H^1(k,G)\big)$$

Note that if G is anisotropic, we have $S=1$ so that $\ker\big(H^1(\mathbb{P}^1_k,G) \xrightarrow{ev_0} H^1(k,G)\big)=1$.

In particular, if a G–torsor over $k[t]$ is trivial at $t=0$ and extends to a G–torsor over $\mathbb{P}^1_k$, then it is trivial. We are given a G-torsor X over $k[t]$ and without loss of generality we can assume that X is trivial on $t=0$. The original method to extend X to the projective line is to use Bruhat-Tits' theory, see [?, ?, ?]. We shall provide a short tricky proof of this extension fact in characteristic zero.

Proof. We assume that k is of characteristic zero. We are given a class $\gamma\in H^1(k[t],G)$ satisfying $\gamma(0)=1$. The idea is to find an integer $d\geq 1$ such that the restriction $\gamma_{k[t^{\frac{1}{d}}]}$ extends to the projective line. This statement is local at ∞ in the sense that it enough to show that there exists d such that $\gamma_{k((t^{-1/d}))}$ comes from $H^1(k,G)$ ([?, 4.1.3], see also [?, cor. A.8]).

According to [?, thm. 1.1], there exists a finite k–subgroup S of G such that $H^1(F,,S)\to H^1(F,G)$ is onto for any k–field F. In particular the map $H^1(k((t^{-1})),S)\to H^1(k((t^{-1})),G)$ is onto.

The absolute Galois group of $k((t^{-1}))$ is $\varprojlim \mu_n(k_s)\rtimes \mathrm{Gal}(k_s/k)=I\rtimes\mathrm{Gal}(k_s/k)$. We are given a cocycle $z: I\rtimes \mathrm{Gal}(k_s/k)\to S(k_s)$, its restriction to the inertia group I is a group homomorphism so factorizes through $\mu_d(k_s)$ for some d. It follows that $[z]_{k((t^{-1/d}))}$ belongs to the image of $H^1(k,S)\to H^1(k((t^{-1/d})),S)$ so that $\gamma_{k((t^{-1/d}))}$ belongs to the image of $H^1(k,G)\to H^1(k((t^{-1/d})),G)$.

We can actually take d to be the order of $S(k_s)$. Our reasoning shows that $\gamma_{k[t^{1/d}]}=1$. By inspection of the proof, one has actually $\gamma_{k[t][\sqrt[d]{at}]}=1$ for any $a\in k^\times$. The trick consists to introduce a new indeterminate u and to extend the setting over $F=k(u)$. We put $\widetilde{\gamma}=\gamma_{k(u)[t][\sqrt[d]{at}]}\in H^1(F[t],G)$ and we have $\widetilde{\gamma}_{k(u)[t][\sqrt[d]{ut}]}=1$. The point is that the fraction field $k(u,t,\sqrt[d]{ut})$ is $k(t)$–isomorphic to $k(t,x)$ (with $x=\sqrt[d]{ut}$) so that

$$\gamma_{k(t)}\in \mathrm{Ker}\Big(H^1(k(t),G)\to H^1(k(t)(x),G)\Big).$$

This kernel is trivial by a specialization argument so that γ is rationally trivial. It follows that γ extends to $\mathbb{P}^1_k$ hence we are done. $\square$

5.4. **The case of the punctered affine line.** This case is more complicated than the affine line.

Theorem 5.8. *(see [?]) Let G be a reductive k–group over a field k of characteristic zero. The map*

$$H^1(k[t^{\pm 1}], G) \xrightarrow{\sim} H^1(k((t)), G)$$

is bijective.

The surjectivity is easy and comes by reduction to a finite subgroup. The hard part is the injectivity where one crucial step is to show an existence of a maximal torus for the relevant twisted group scheme. This involves Bruhat-Tits theory and twin buildings.

Bruhat-Tits theory also provides a description of $H^1(k((t)), G)$ [?].

6. What is next?

Quillen-Suslin and al: that is, Raghunathan's results [?], (resp. Asok-Hoyois-Wendt [?]) over polynomial rings, Stavrova's results on Laurent polynomial rings [?].

Fedorov exotic examples of non constant G–torsor over $R[t]$ with R local ring [?]. This involves patching and affine Grassmannians.

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