Classifying spaces in motivic homotopy theory (4 lectures)

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These lectures will study the classifying space BG of an algebraic group G, a basic example in motivic homotopy theory. I will discuss how to compute the Chow groups of algebraic cycles, a motivic analog of homology groups. The focus will be on the Chow ring CH^*BG and more generally the equivariant Chow ring $CH^*_G X$ for a smooth scheme X with an action of G. Two ways to look at this are:

(1) Sometimes the equivariant Chow ring $CH^*_G X$ and equivariant cohomology $H^*_G(X, \mathbf{Z})$ can both be used for a geometric problem, and you can take your choice. There may be technical advantages to $CH^*_G X$.

(2) Sometimes the difference between $CH^*_G X$ and $H^*_G(X, \mathbf{Z})$ is interesting, as a way to measure the difference between algebraic geometry and topology.

Some references on Chow groups are: W. Fulton, *Intersection theory* [4], the main reference book; D. Eisenbud and J. Harris, *3264 and all that* [2], a gentler introduction; and the notes from Ravi Vakil's course on Intersection Theory on the web [10].

Some references on classifying spaces in algebraic geometry are Morel-Voevodsky's "A^1-homotopy theory of schemes" [6], my paper "The Chow group of a classifying space" [7], my book *Group Cohomology and Algebraic Cycles* [8], and Edidin-Graham's "Equivariant intersection theory" [1].

Chapter 1

Lecture 1

1.1 Quick introduction to classifying spaces in topology

Classifying spaces in topology go back to the birth of group cohomology in the 1940s (Eckmann, Eilenberg, MacLane, Steenrod). The classifying space (in the sense we consider) was introduced into algebraic geometry by Morel-Voevodsky [6] and Totaro [7].

In topology, for a topological group G (which could be a discrete group), the *classifying space* BG means the quotient space EG/G, where EG is any contractible space with a free G-action. Under weak point-set topological assumptions, there always is such a space EG, and the homotopy type of BG is independent of the choice of EG. By the fibration $G \to EG \to BG$, G is homotopy equivalent to the loop space of BG:

$$\Omega BG \simeq G.$$

The constructions $X \mapsto \Omega X$ and $G \mapsto BG$ give equivalences between the homotopy category of connected pointed topological spaces and the homotopy category of topological groups (or, perhaps more naturally, topological monoids M with $\pi_0 M$ a group).

For example, when G is a discrete group, we can define group cohomology by the confusing-looking definition:

$$H^*(G, \mathbf{Z}) := H^*(BG, \mathbf{Z}).$$

For a discrete group G, another name for BG is a $K(G, 1)$ space.

A *principal G-bundle* over a topological space X is a space E with a free action of G and an identification of E/G with X:

$$E \underset{G}{\to} X.$$

One should also assume here that the map $\pi \colon E \to X$ is locally a product, in other words that X is covered by open sets U for which $\pi^{-1}(U) \cong G \times U$, with G acting on $G \times U$ by left translation on the G factor. (Under weak point-set topological assumptions, this local triviality is automatic.)

Example: For reasonable spaces X, there is an equivalence of categories (with all morphisms taken to be isomorphisms) between the real vector bundles of rank n over a space X and the principal $GL(n, \mathbf{R})$-bundles over X. (This helps to motivate the study of principal G-bundles.) In one direction: send a principal $GL(n, \mathbf{R})$-bundle $E \to X$ to the rank-n vector bundle

$$(E \times \mathbf{R}^n)/GL(n, \mathbf{R}) \to E/GL(n, \mathbf{R}) = X,$$

where $GL(n, \mathbf{R})$ acts "diagonally" on E and on $\mathbf{R}^n$.

Likewise, complex vector bundles of rank n are equivalent to principal $GL(n, \mathbf{C})$-bundles. And principal G-bundles for a subgroup G of $GL(n)$ correspond to vector bundles with "extra structure". For example, principal bundles for the unitary group $U(n)$ correspond to complex vector bundles of rank n with a hermitian metric. In this example, the inclusion $U(n) \to GL(n, \mathbf{C})$ is a homotopy equivalence, and so complex vector bundles of rank n *up to isomorphism* can be identified with principal $U(n)$-bundles up to isomorphism.

A strong motivation for studying classifying spaces in topology is that they classify principal G-bundles. Namely, for any space X, let $H^1(X, G)$ be the set of isomorphism classes of principal G-bundles over X. (This is not a group, unless G is abelian, but only a pointed set. The base point of $H^1(X, G)$ is the trivial G-bundle over X, $G \times X \to X$.) Then we have (for reasonable spaces X):

$$H^1(X, G) \cong [X, BG],$$

the set of (unbased) homotopy classes of continuous maps from X to BG. In one direction: $EG \to BG$ is a principal G-bundle over BG, the "universal bundle", and any continuous map $f\colon X \to BG$ determines a principal G-bundle over X by pulling back:

$$\begin{tikzcd} X \times_{BG} EG \arrow[r] \arrow[d, "G"] & EG \arrow[d, "G"] \\ X \arrow[r] & BG. \end{tikzcd}$$

1.2 The classifying space in algebraic geometry

Let G be an affine group scheme of finite type over a field k. To explain some of these words: an *affine scheme of finite type over a field* means a closed subscheme of affine n-space A^n_k for some $n \geq 0$. And the closed subschemes of A^n_k are in one-to-one correspondence with the *ideals* in the polynomial ring $k[x_1, \ldots, x_n]$. (The *ring of regular functions* on A^n_k, $O(A^n_k)$, means $k[x_1, \ldots, x_n]$.) Given some polynomials $f_1, \ldots, f_r$ in $k[x_1, \ldots, x_n]$, our notation for a closed subscheme is

$$\{f_1 = 0, \ldots, f_r = 0\} \subset A^n_k,$$

meaning the scheme $\operatorname{Spec} k[x_1, \ldots, x_n]/(f_1, \ldots, f_r)$. We define a scheme X (such as this one) as a topological space with a sheaf of commutative rings, called the sheaf of *regular functions* O_X. We use the Zariski topology, where the only open sets are the complements of closed *algebraic* subsets.

A *group scheme* over a field k is a scheme G over k with morphisms $G \times_k G \to G$, $\operatorname{Spec} k \to G$, and $G \to G$ (multiplication, identity, inverse) that satisfy the axioms of a group.

Example 1.2.1. Every finite group G determines an affine group scheme of finite type over k: just take the union of one copy of $\operatorname{Spec} k$ for each element of G. That is a group scheme of dimension zero. A basic example of a positive-dimensional group is the group scheme $GL(n)$ over k, the group of invertible $n \times n$ matrices. Some other important examples of affine group schemes are $SL(n)$ (the matrices with determinant 1), $O(n)$ (or more accurately $O(q)$), the subgroup of $GL(n)$ preserving a nondegenerate quadratic form q of dimension n over k, or $Sp(2n)$, the subgroup of $GL(2n)$ preserving a nondegenerate alternating bilinear form. Some more exotic examples are the exceptional groups: G_2, F_4, E_6, E_7, E_8. Some simple examples are the additive and multiplicative groups over k, $G_a = A^1_k$ and $G_m = A^1_k - 0$.

In topology, BG is usually infinite-dimensional. For example, $BS^1 \simeq \mathbf{CP}^\infty$, and so

$$H^*(BS^1, \mathbf{Z}) = \mathbf{Z}[u]$$

with $|u| = 2$, which shows that there is no finite-dimensional model for BS^1. (Equivalently, S^1 cannot act freely on a finite-dimensional contractible space; with more care, you can show that every action of S^1 on a finite-dimensional contractible space has a fixed point.) Since algebraic varieties are finite-dimensional, it is not clear how to construct classifying spaces in algebraic geomety.

The basic idea is that every affine group scheme G of finite type over a field k has a finite-dimensional faithful representation V, $G \hookrightarrow GL(V)$. Here G will act freely on V outside some Zariski-closed subset $S \subset V$. Then we think of $(V - S)/G$ as a finite-dimensional approximation to BG. The key point is that we can find representations V with the codimension of S in V as large as we like.

In Morel-Voevodsky's motivic homotopy category over the field k, one can define

$$BG = \mathrm{colim}_{\mathrm{codim}(S \subset V) \to \infty} (V - S)/G.$$

This is independent of the choice of representations, up to A^1-homotopy equivalence. (One can arrange to have a sequence of morphisms between the quotient schemes $(V - S)/G$.) Morel and Voevodsky call this the *étale classifying space*, $B_{\text{ét}}G$ [6, p. 130].

Example 1.2.2. Over any field k, let the multiplicative group G_m act on A^{n+1}_k by

$$t(x_0, \ldots, x_n) = (tx_0, \ldots, tx_n).$$

Then

$$BG_m = \mathrm{colim}_n (A^{n+1} - 0)/G_m = \mathrm{colim}_n \mathbf{P}^n.$$

Thus, informally speaking, BG_m is $\mathbf{P}^\infty$ over k.

When k is the complex numbers $\mathbf{C}$, we have the complex realization functor from the A^1-homotopy category to the usual homotopy category, taking a scheme X over $\mathbf{C}$ to its space $X(\mathbf{C})$ of complex points with the classical topology. By construction, the complex realization of BG is the topological classifying space $B(G(\mathbf{C}))$. (The assumption that $\mathrm{codim}(S \subset V) \to \infty$ ensures that the open sets $V - S$ get closer and closer to being contractible.) In particular, the cohomology of the motivic classifying space is the same as the cohomology of classifying spaces in topology.

But we have gained something by realizing the classifying space in algebraic geometry. For example, the *Chow ring* CH^*BG is a new invariant of G (even when G is just a finite group), with a homomorphism (when $k = \mathbf{C}$)

$$CH^*BG \to H^*(BG, \mathbf{Z})$$

that is *not* always an isomorphism.

1.3 Chow groups

We give here a quick introduction to Chow groups. In short, these are an analog of homology groups for algebraic varieties, generated only by algebraic subvarieties. In a sense, the difference between Chow groups and ordinary homology measures the difference between algebraic geometry and topology.

Chow groups were defined by Francesco Severi in the 1930s. The main improvements to the construction were made by Wei-Liang Chow in the 1950s and by William Fulton and Robert MacPherson in the 1970s. I will give very few details; see the references at the start of this lecture.

Throughout, we fix a field k, and a *scheme* will mean a separated scheme of finite type over k. (It would be fine to restrict to quasi-projective schemes over k.) A *variety* means an integral separated scheme of finite type over k. (In particular, a variety is irreducible, by this definition.)

Definition 1.3.1. The group Z_*X of *algebraic cycles* on a scheme X over k (as above) is the free abelian group on the set of closed subvarieties of X. This is graded by dimension, $Z_*X = \oplus_i Z_i X$.

Definition 1.3.2. For a variety X over k, a *divisor* on X means an algebraic cycle of codimension 1. That is, for $n = \dim X$, a divisor is a formal finite sum $\sum a_i D_i$ with a_i integers and $D_i \subset X$ subvarieties of dimension $n-1$. (Sometimes we write brackets around subvarieties: $\sum a_i[D_i]$.) An *irreducible divisor* on X means a subvariety of codimension 1.

Definition 1.3.3. For a rational function f on a variety X, not identically zero, and an irreducible divisor D on X,

$$\mathrm{ord}_D(f)$$

denotes the order of vanishing of f along D, which is an integer (negative if f has a pole along D). See Fulton or Eisenbud-Harris for details.

We write $k(X)$ for the field of rational functions on a variety X. For a rational function $f \in k(X)^*$, the *divisor of* f is

$$(f) := \sum \mathrm{ord}_D(f)[D],$$

where the sum runs over all codimension-1 subvarieties D of X. Here (f) is a finite sum, as our definition of divisor requires. (That is, f only has zeros and poles on finitely many codimension-1 subvarieties.)

Definition 1.3.4. Two i-cycles on a scheme X over k are said to be *rationally equivalent* if their difference lies in the subgroup generated by all divisors (f) of all rational functions $f \in k(W)^*$ for all $(i+1)$-dimensional subvarieties W of X. (Notice that we can view (f) as an i-cycle on X, in this case.) The (i-dimensional) *Chow group* CH_iX is the abelian group of i-cycles on X modulo rational equivalence.

Thus, to find generators of Chow groups from the definition, it seems that you would have to know *all* subvarieties of X. The relations are even more complicated. This might make it seem that there is no hope of computing Chow groups. And indeed, the Chow groups are unknown for many varieties; they are far more mysterious than homology groups, in general. Fortunately, there are many varieties for which we can compute Chow groups, and whenever we can do that, it says a lot. Moreover, the formal properties of Chow groups are very good, which makes it possible to pass information from one variety to another.

Example 1.3.5. For a scheme X of dimension n, CH_iX is zero unless $0 \leq i \leq n$. (That should be clear from the definition.)

For a variety X of dimension n, $CH_nX \cong \mathbf{Z}$, generated by the class $[X]$. (Again, I hope this is clear from thinking through the definition: X is the only n-dimensional subvariety of X, and there are no $(n+1)$-dimensional subvarieties.) Also, $CH_{n-1}X$ is the *divisor class group* $\mathrm{Cl}\, X$, the group of divisors on X modulo linear equivalence (that is, modulo divisors of rational functions on X).

For any scheme X, the Chow groups of X are the same as the Chow groups of the underlying reduced scheme $X_{\text{red}} \subset X$, since the subvarieties of X_{red} are the same as the subvarieties of X.

Some basic calculations, proved in the references: the Chow groups of affine space A^n_k are

$$CH_iA^n_k \cong \begin{cases} \mathbf{Z} & \text{if } i = n \\ 0 & \text{otherwise.} \end{cases}$$

The Chow groups of projective space $\mathbf{P}^n_k$ are

$$CH_i\mathbf{P}^n_k \cong \begin{cases} \mathbf{Z} & \text{if } 0 \leq i \leq n \\ 0 & \text{otherwise.} \end{cases}$$

For $0 \leq i \leq n$, the ith Chow group of $\mathbf{P}^n$ is generated by the class of any i-dimensional linear subspace over k, $\mathbf{P}^i_k \subset \mathbf{P}^n_k$.

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For $k = \mathbf{C}$, we have a natural homomorphism, the *cycle map*:

$$CH_iX \to H^{BM}_{2i}(X, \mathbf{Z}).$$

The groups on the right are the *Borel-Moore homology groups*, which depend only on the space $X(\mathbf{C})$ with the classical topology (unlike Chow groups). In short, these are the homology of a locally compact space "relative to infinity". So $H^{BM}_j(X, \mathbf{Z})$ is the usual homology group $H_j(X, \mathbf{Z})$ for X compact. More generally, for a closed subset Y of a compact space X, the Borel-Moore homology of $X - Y$ is the relative homology of the pair (X, Y):

$$H^{BM}_j(X - Y, \mathbf{Z}) \cong H_j(X, Y; \mathbf{Z}).$$

Borel-Moore homology comes up because we are not assuming that X is proper over $\mathbf{C}$, i.e., that $X(\mathbf{C})$ is compact in the classical topology. So closed subvarieties of X also need not be compact. A noncompact manifold does not have a fundamental class in ordinary homology, but it does in Borel-Moore homology. As a result, Poincaré duality for noncompact manifolds is expressed in terms of Borel-Moore homology: for an oriented real n-manifold X,

$$H^i(X, \mathbf{Z}) \cong H^{BM}_{n-i}(X, \mathbf{Z}).$$

Every complex manifold has a natural orientation (as a real manifold), which explains why a *smooth* i-dimensional subvariety Y of a complex scheme X has a class in $H_{2i}^{BM}(X, \mathbf{Z})$. Even if Y is singular, it has a fundamental class in $H_{2i}^{BM}(Y, \mathbf{Z})$ and hence in $H_{2i}^{BM}(X, \mathbf{Z})$. In short, this is because the singular set of Y has complex codimension at least 1, hence real codimension at least 2, which does not affect the homology in the top dimension. Fulton shows that two rationally equivalent cycles are homologous, and so we have the cycle map

$$CH_iX \to H_{2i}^{BM}(X, \mathbf{Z}).$$

Definition 1.3.6. For a smooth scheme X of dimension n over a field, we can also number Chow groups by codimension:

$$CH^iX := CH_{n-i}X.$$

Thus, for X smooth over $\mathbf{C}$, we can rewrite the cycle map (using Poincaré duality) as

$$CH^iX \to H^{2i}(X, \mathbf{Z}).$$

(To remember the numbering: a subvariety of complex codimension i in X has real codimension $2i$.) To avoid confusion, use the notation CH^iX only when X is smooth over k.

For a smooth scheme X over a field k, the Chow groups (numbered by codimension) form a commutative graded ring:

$$CH^iX \times CH^jX \to CH^{i+j}X.$$

In short, the product describes the *intersection* of algebraic cycles, although it takes some effort to get the details right. For X smooth over $\mathbf{C}$, the cycle map is a ring homomorphism from CH^*X to $H^*(X, \mathbf{Z})$. (Indeed, for cohomology classes represented by submanifolds of X, the cup product also corresponds to intersecting the submanifolds, when the intersection is transverse.)

For example, using that the intersection of two linear subspaces in $\mathbf{P}^n$ is a linear subspace, the Chow ring of $\mathbf{P}^n$ (over any field k) is given by

$$CH^*\mathbf{P}^n = \mathbf{Z}[u]/(u^{n+1}).$$

Here u is the class of a hyperplane $\mathbf{P}^{n-1} \subset \mathbf{P}^n$ over k, and u^i is the class of a linear subspace of codimension i, for $0 \leq i \leq n$.

Chapter 2

Lecture 2

Today, I will present some general constructions with Chow groups, and then start studying the Chow ring of a classifying space BG.

2.1 Constructions with Chow groups

2.1.1 The class of a closed subscheme

Let X be a scheme over a field k (X is separated and of finite type over k, by our conventions). Let $Y \subset X$ be a closed subscheme of dimension r. (So Y might have irreducible components of different dimensions, but the largest dimension is r.) Then Y determines a well-defined cycle

$$[Y] \in Z_r X,$$

and hence an element of $CH_r X$. The construction is: look at the r-dimensional irreducible components of Y, say $S_1, \ldots, S_r$ (and ignore any lower-dimensional components of Y). We will have

$$[Y] = \sum_i a_i [S_i] \in Z_r X$$

for some positive integers a_i. The problem is to define the number a_i, called the *multiplicity* of Y along S_i.

The idea is to look at the local ring O_{X,S_i}. By definition, an element of O_{X,S_i} is a regular function on some open subset of X that has nonempty intersection with S_i. (By the properties of the Zariski topology, an element of O_{X,S_i} restricts to a regular function on most of S_i, in fact on all of S_i outside a lower-dimensional subset.) Working with this ring is a way of ignoring anything that happens on lower-dimensional subsets of S_i (or anything away from S_i).

The prime ideals in O_{X,S_i} are in one-to-one correspondence with the subvarieties of X *that contain* S_i. As a result, the dimension of the ring O_{X,S_i} (the maximum length of chains of prime ideals) is

$$\dim O_{X,S_i} = \operatorname{codim}(S_i \subset X).$$

Let I_Y be the ideal of the closed subscheme Y in O_{X,S_i}, that is, the ideal of functions that vanish on Y near S_i. Then

$$\dim O_{X,S_i}/I_Y = 0,$$

because S_i and Y have the same dimension.

It follows that, as an O_{X,S_i}-module, $O_{X,S_i}/I_Y$ is an extension of finitely many copies of the residue field $O_{X,S_i}/\mathfrak{m} = k(S_i)$. We define the multiplicity a_i of S_i in Y as the number of copies of $O_{X,S_i}/\mathfrak{m}$ that occur in a composition series for $O_{X,S_i}/I_Y$. (This is well-defined, by the Jordan-Hölder theorem.)

Example 2.1.1. For the closed subscheme $Y = \{x^2 y = 0\} \subset A^2_k$, the associated 1-cycle on A^2_k is

$$[Y] = 2 \cdot \{x = 0\} + 1 \cdot \{y = 0\}.$$

2.1.2 Chern classes

For an (algebraic) vector bundle E on a variety X of dimension n, we have its *Chern classes* $c_i E$ in $CH_{n-i}X$. I will concentrate on the case where X is smooth over k, in which case $c_i E$ is in $CH^i X$. These have the same geometric interpretations as in topology: roughly speaking, $c_i E$ is the first obstruction for E to have $n - i + 1$ linearly independent sections. (See section 2.1.9 for an explicit definition of Chern classes.) As in topology, we have $c_0 E = 1 \in CH^0 X$ and $c_i E = 0$ for $i > \text{rank}(E)$.

Chern classes in the Chow ring have the same formal properties as in topology. In particular, for an exact sequence

$$0 \to A \to B \to C \to 0$$

of vector bundles on a smooth k-scheme X, the *total Chern class* $c(B) = 1 + c_1(B) + c_2(B) + \cdots$ in $CH^* X$ satisfies

$$c(B) = c(A)c(C).$$

In topology, an exact sequence of vector bundles on a reasonable space always splits. In algebraic geometry, that is not true, but we still have the formula above for Chern classes.

Example 2.1.2. For a vector bundle E of rank r, the top Chern class $c_r E$ in $CH^r X$ is the class of the zero scheme of a global section $s \in H^0(X, E)$, *if* there is a section whose zero set has codimension r. (That would always be true in topology, but not always in algebraic geometry. The basic obstacle is that this construction always gives a cycle with nonnegative coefficients, whereas in general, one may need a cycle with negative coefficients to represent $c_r(E)$.)

For a line bundle L on a variety X, a variant of this definition always works. Namely, let s be a *rational* section of L, not identically zero. (That is, s is a regular section on some nonempty open subset of X; this clearly exists, because L is trivial on some nonempty open subset of X.) Then we can define the first Chern class of L as the class of the divisor (s) in $CH_{n-1}X$. Here (s) measures the zeros and poles of s, as in the definition of the divisor of a rational function. Note that (s) is *not* necessarily linearly equivalent to zero, because L is locally trivial but not necessarily globally trivial.

We can view the first Chern class on an n-dimensional variety X as a group homomorphism

$$c_1 \colon \text{Pic}\, X \to CH_{n-1}X = \text{Cl}\, X.$$

Here the *Picard group* $\operatorname{Pic} X$ is the abelian group of isomorphism classes of line bundles on X, with the group operation being tensor product. If X is smooth over k, then the first Chern class is in fact an isomorphism:

$$c_1 \colon \operatorname{Pic} X \xrightarrow{\cong} CH^1 X.$$

To go backwards from a divisor D on X to a line bundle, define a sheaf of O_X-modules $O(D)$ by:

$$O(D)(U) = \{f \in k(X) : (f) + D \geq 0 \text{ on } U\}$$

for nonempty open subsets $U \subset X$. When X is smooth, the sheaf $O(D)$ is a line bundle, using that the local rings of X are unique factorization domains.

2.1.3 Proper pushforward

For a proper morphism $f \colon X \to Y$ of k-schemes, Fulton defines a homomorphism called *proper pushforward*:

$$f_* \colon CH_i X \to CH_i Y.$$

In short, for an i-dimensional subvariety $S \subset X$, let T be its image in Y. We define $f_*[S] = 0$ if T has dimension less than i. Otherwise,

$$f_*[S] = \deg(S \to T)[T].$$

Here the *degree* of a dominant morphism $S \to T$ of varieties of the same dimension can be defined as the degree of the function field $k(S)$ as an extension of $k(T)$. (This agrees with the topological notion of degree when the base field is $\mathbf{C}$.) A morphism is *dominant* if the image is dense.

To see that we cannot expect a pushforward homomorphism on Chow groups for a non-proper morphism, consider the inclusion of the affine line as an open subscheme of the projective line, $f \colon A^1_k \to \mathbf{P}^1_k$. The only reasonable way to define the pushforward of the point $0 \in A^1$ would be to set $f_*[0] = [0]$. But $[0]$ is rationally equivalent to zero on A^1 (because it is the divisor of the rational function $x \in O(A^1) = k[x]$), but $[0]$ is not rationally equivalent to zero on $\mathbf{P}^1_k$. (In fact, $[0]$ is a generator of the group $CH_0(\mathbf{P}^1_k) \cong \mathbf{Z}$.) So the pushforward homomorphism at the level of cycles, $f_* \colon Z_0 A^1 \to Z_0 \mathbf{P}^1$, does not pass to a well-defined homomorphism $CH_0 A^1 \to CH_0 \mathbf{P}^1$. (We can view the function x as a rational function on $\mathbf{P}^1$, but it vanishes at 0 and also has a pole at ∞; so it only shows that $[0] = [\infty]$ in $CH_0 \mathbf{P}^1_k$.)

This is also to be expected from the properties of Borel-Moore homology groups. The usual homology groups are functorial for all continuous maps, but Borel-Moore homology groups only have a pushforward for proper morphisms (of locally compact spaces).

2.1.4 Degree of a zero-cycle

Let X be a proper scheme over a field k. That is, we are given a proper morphism $X \to \operatorname{Spec} k$. Applying proper pushforward gives a homomorphism called the *degree*:

$$\deg \colon CH_0 X \to CH_0(\operatorname{Spec} k) = \mathbf{Z}.$$

We can describe this homomorphism explicitly, from the definition from proper pushforward. Namely, CH_0X is generated by the zero-dimensional subvarieties of X, or equivalently, the closed points in the scheme X. (A scheme is, in particular, a topological space, and in general, not every point is closed.) Each closed point of X is isomorphic to $\operatorname{Spec} E$ for some finite extension field E of k, and we have

$$\deg(\operatorname{Spec} E) = [E : k],$$

the degree of E over k (that is, the dimension of E as a k-vector space).

For example, let X be the projective line $\mathbf{P}^1$ over the real numbers $\mathbf{R}$. Let x be the coordinate function on the affine line $A^1_{\mathbf{R}} \subset \mathbf{P}^1_{\mathbf{R}}$. Then, for example, $x = 0$ defines a closed point P of degree 1 in $\mathbf{P}^1_{\mathbf{R}}$, while $x^2+1 = 0$ defines a closed point Q of degree 2 (isomorphic to $\operatorname{Spec} \mathbf{R}[x]/(x^2+1) \cong \operatorname{Spec} \mathbf{C}$). For $\mathbf{P}^1$ over any field k, the degree homomorphism $\deg\colon CH_0(\mathbf{P}^1_k) \to \mathbf{Z}$ is an isomorphism, and so $[Q]$ should be linearly equivalent to $2[P]$. Sure enough, the divisor of the rational function $x^2/(x^2 + 1)$ is $2[P] - [Q]$. (Note that this function has neither a zero nor a pole at the point $\infty = \mathbf{P}^1 - A^1$.)

2.1.5 Chow groups of a smooth projective curve

Let X be a smooth projective curve over a field k. For convenience, assume that X has a k-rational point. Then $CH^0X \cong \mathbf{Z}$, and so the only interesting Chow group is $CH^1X \cong \operatorname{Pic} X$, the Chow group of zero-cycles. A central result of algebraic geometry is the calculation:

$$0 \to \operatorname{Jac}(X)(k) \to CH^1X \to \mathbf{Z} \to 0,$$

where $CH^1(X) \to \mathbf{Z}$ is the degree of a zero-cycle. Here $\operatorname{Jac}(X)$ is the *Jacobian* of X, an abelian variety of dimension equal to the genus g of X. (This is a commutative group scheme over k, and we write $\operatorname{Jac}(X)(k)$ for its group of k-rational points.)

For $k = \mathbf{C}$, we can identify the group of complex points of the Jacobian with the torus $(S^1)^{2g}$. In particular, the sequence above shows that CH^1X is an *uncountable* abelian group for a curve of genus at least 1 over $\mathbf{C}$. As a result, it is not clear what it would even mean to compute the Chow groups of a complex variety in general.

2.1.6 Flat pullback and the localization sequence

A morphism of schemes, $f\colon X \to Y$, is called *flat* if, on affine charts in X and Y, it corresponds to a flat ring homomorphism $A \to B$, meaning that B is flat as an A-module. Informally, the fibers of a flat morphism form a well-behaved "family" of schemes; in particular, all the nonempty fibers have the same dimension (if Y is connected).

For a flat morphism $f\colon X \to Y$ of relative dimension r, we have a pullback homomorphism (*flat pullback*):

$$f^*\colon CH_iY \to CH_{i+r}X.$$

Like proper pushforward, this comes from an operation at the level of cycles. Namely, for a subvariety $S \subset Y$, we define

$$f^*[S] = [f^{-1}(S)],$$

meaning the class of the closed subscheme $f^{-1}(S) \subset X$. (This was one reason for presenting the construction of the cycle associated to a closed subscheme.)

Example 2.1.3. For an open subscheme $U \subset X$, the inclusion $f\colon U \hookrightarrow X$ is flat of relative dimension 0, and so we have a *restriction* homomorphism

$$f^*\colon CH^iX \to CH^iU.$$

Proper pushforward and flat pullback both appear in the *localization sequence* for Chow groups, a key computational tool. Namely, for a closed subscheme Z of a scheme X over k, we have an exact sequence:

$$CH_iZ \to CH_iX \to CH_i(X-Z) \to 0.$$

The exactness on the right is a distinctive feature of Chow groups. Geometrically, this happens because for any closed subvariety of $X - Z$, its closure in X is a closed subvariety of X.

For $k = \mathbf{C}$, the localization sequence for Chow groups maps to the localization sequence for Borel-Moore homology, but note the differences:

$$\cdots \to H^{BM}_{2i+1}(X-Z) \to H^{BM}_{2i}Z \to H^{BM}_{2i}X \to H^{BM}_{2i}(X-Z) \to H^{BM}_{2i-1}Z \to \cdots.$$

For example, the localization sequence for Chow groups implies that for any Zariski open subset U of affine space A^n, we have $CH^iU = 0$ for all $i \neq 0$, whereas (for $k = \mathbf{C}$) the cohomology of U can be big and complicated (depending on what you remove from A^n).

A big difficulty with the localization sequence for Chow groups is that it says nothing about the kernel of $CH_iZ \to CH_iX$. That was a central motivation for the extension of Chow groups to *motivic cohomology* (or "Borel-Moore motivic homology"), by Bloch and Voevodsky. Namely, one can extend the localization sequence to the left, but it involves these more general invariants, not just Chow groups. Namely, motivic cohomology is a bigraded abelian group, $H^i(X, \mathbf{Z}(j))$, and Chow groups (for a smooth k-scheme X) are the special case

$$CH^iX = H^{2i}(X, \mathbf{Z}(i)).$$

For convenience, let us write out the localization sequence with cycles indexed by codimension, when Z is a smooth codimension-r subscheme of a smooth scheme X over k:

$$CH^{i-r}Z \to CH^iX \to CH^i(X-Z) \to 0.$$

2.1.7 Homotopy invariance for Chow groups

Let $f\colon X \to Y$ be an A^r-bundle. By this, I mean that Y is covered by open subsets U such that $f^{-1}(U)$ is isomorphic to $U \times A^r$ over U. (I am not assuming anything about the structure group of this fibration; so this is more general than the case of a vector bundle.) Then the pullback homomorphism on Chow groups is an isomorphism:

$$f^*\colon CH_iY \xrightarrow{\cong} CH_{i+r}X.$$

If Y is smooth over k, then we can write this more neatly as:

$$f^*\colon CH^iY \xrightarrow{\cong} CH^iX.$$

This is known as *homotopy invariance for Chow groups.*

Let L be a line bundle over a smooth scheme X over k. Algebraic geometers traditionally view L as a sheaf of O_X-modules, but we can also view it as a scheme with a morphism $L \to X$ (and fibers isomorphic to A^1). Using homotopy invariance for Chow groups plus the localization sequence, we can compute the Chow ring of L minus the zero section (isomorphic to X):

$$CH^*(L - X) \cong CH^*X/(c_1L).$$

2.1.8 Pullback for smooth schemes

Let $f\colon X \to Y$ be *any* morphism of smooth schemes over k. Then there is a pullback homomorphism

$$f^*\colon CH^iY \to CH^iX.$$

In fact, this is a homomorphism of graded rings from CH^*Y to CH^*X. It agrees with flat pullback if f happens to be flat. In general, this is nontrivial to define, much like the intersection product on Chow groups. In the special case of a subvariety $Z \subset Y$ whose inverse image $f^{-1}(Z) \subset X$ has the same codimension, we can say that

$$f^*[Z] = [f^{-1}(Z)],$$

the class of the subscheme $f^{-1}(Z)$ in CH^*X.

As you might expect, for smooth schemes over $\mathbf{C}$, the pullback on Chow rings is compatible with the pullback on cohomology:

$$\begin{tikzcd} CH^iY \arrow[r] \arrow[d] & CH^iX \arrow[d] \\ H^{2i}(Y,\mathbf{Z}) \arrow[r] & H^{2i}(X,\mathbf{Z}). \end{tikzcd}$$

2.1.9 Projective bundle theorem

Generalizing the calculation of the Chow groups of projective space, we describe here the Chow groups of a projective bundle. To simplify the statement, we assume that the base space is smooth, although that is not necessary.

Namely, let E be a vector bundle of rank r on a smooth k-scheme X. Following Fulton's notation (not Grothendieck's), let $\pi\colon P(E) \to X$ be the associated projective bundle, the space of lines in the fibers of E. (Thus π is a $\mathbf{P}^{r-1}$-bundle.) There is a natural line bundle $O(-1)$ on $P(E)$ whose fiber at a point is the corresponding line in E. Thus we have an exact sequence of vector bundles on $P(E)$,

$$0 \to O(-1) \to \pi^*E \to Q \to 0,$$

where Q has rank $r-1$. Let $v = c_1O(-1)$.

Theorem 2.1.4. *The Chow ring $CH^*P(E)$ is a free module over CH^*X with basis $1, v, \ldots, v^{r-1}$.*

It follows that $v^r \in CH^r P(E)$ must be some linear combination of $1, v, \ldots, v^{r-1}$ with coefficients in $CH^* X$. In fact, the coefficients are exactly the Chern classes of X, up to sign. This is Grothendieck's way of *defining* the Chern classes of E. It works in topology as well. Namely, we have:

$$c_r(E) - vc_{r-1}(E) + \cdots + (-v)^{r-1}c_1(E) + (-v)^r = 0$$

in $CH^* P(E)$.

2.2 The Chow ring of a classifying space

Definition 2.2.1. Let G be an affine group scheme of finite type over a field k. For $i \geq 0$, we can define $CH^i BG$ as follows. Let V be a finite-dimensional representation of G over k, and view V as a scheme (namely, affine space of some dimension over k). Let S be a closed G-invariant subset of V. If S has codimension greater than i in V, then we define

$$CH^i BG = CH^i((V - S)/G).$$

We will show that this is independent of the choice of V and S. First note that we can find pairs (V, S) as above with the codimension of S in V as large as we like. Namely, use the fact that G has a faithful representation W over k. Then you can check that G acts freely on the direct sum $W^{\oplus N}$ outside a closed subset whose codimension goes to infinity as N goes to infinity.

To prove that $CH^i BG$ is well-defined, consider two pairs (V_1, S_1) and (V_2, S_2) as above. That is, V_1 and V_2 are finite-dimensional representations of G, and $S_1 \subset V_1$ and $S_2 \subset V_2$ are closed subsets of codimension $> i$ such that G acts freely on $V_1 - S_1$ and on $V_2 - S_2$. We want to construct an isomorphism

$$CH^i((V_1 - S_1)/G) \cong CH^i((V_2 - S_2)/G).$$

(To be precise, we should also check that this isomorphism is independent of choices; we will not bother with that point here.)

The idea is to compare both V_1 and V_2 to the direct sum $V_1 \oplus V_2$. Indeed, we have morphisms

$$((V_1 - S_1) \times V_2)/G \to (V_1 - S_1)/G$$

and

$$(V_1 \times (V_2 - S_2))/G \to (V_2 - S_2)/G.$$

Moreover, these are both vector bundles (with fiber V_2 or V_1, respectively). To be precise, these are manifestly vector bundles for the flat topology; but, by Hilbert's Theorem 90 or Grothendieck's theory of faithfully flat descent, vector bundles in the flat topology are the same as the usual notion of vector bundles in the Zariski topology.

As a result, homotopy invariance of Chow rings gives that both morphisms above induce isomorphisms of Chow rings. It remains to show that $((V_1 - S_1) \times V_2)/G$ and $(V_1 \times (V_2 - S_2))/G$ have isomorphic Chow rings. To do that, compare both varieties to

$$((V_1 - S_1) \times (V_2 - S_2))/G,$$

which is an open subset of both of them. The point is that, by our assumption on S_1 and S_2, we are removing subsets of codimension greater than i, in both cases. Therefore, the localization sequence gives that

$$\begin{aligned} CH^i(((V_1 - S_1) \times V_2)/G) &\cong CH^i(((V_1 - S_1) \times (V_2 - S_2))/G) \\ &\cong CH^i((V_1 \times (V_2 - S_2))/G). \end{aligned}$$

This completes the proof that CH^iBG is well-defined.

Moreover, the varieties $(V-S)/G$ that we use to approximate BG are all smooth, and so we can talk about their Chow rings. By inspection of the proof above, the Chow rings of $(V_1 - S_1)/G$ and $(V_2 - S_2)/G$ agree in degrees at most i. Therefore, we have a well-defined commutative graded ring CH^*BG which agrees with the Chow ring of $(V - S)/G$ in degrees at most i, for any i and any pair (V, S) as above with $\operatorname{codim}(S \subset V) > i$.

Example 2.2.2. Consider the multiplicative group G_m over a field k. Then G_m has an obvious faithful representation W of dimension 1, $G_m \xrightarrow{\cong} GL(1)$. Taking the direct sum of $N + 1$ copies of W gives a representation on which G_m acts freely outside the origin:

$$t(x_0, \ldots, x_n) = (tx_0, \ldots, tx_n).$$

Therefore, for $0 \leq i \leq N$, the definition of CH^iBG_m gives that

$$\begin{aligned} CH^iBG_m &\cong CH^i((A^{N+1} - 0)/G_m) \\ &\cong CH^i\mathbf{P}^N \\ &\cong \mathbf{Z}. \end{aligned}$$

Moreover, the Chow ring of BG_m is defined to agree with the Chow ring of $\mathbf{P}^N$ in degrees at most N, and so we have

$$CH^*BG_m \cong \mathbf{Z}[u]$$

with $|u| = 1$ (meaning that u is in CH^1BG_m). (Here we define CH^*BG as the direct sum of the groups CH^iBG, but you might prefer to consider the direct product, which would be the power series ring $\mathbf{Z}[[u]]$.)

Example 2.2.3. Let G be the cyclic group $\mathbf{Z}/m = \langle \sigma : \sigma^m = 1 \rangle$, for a positive integer m. Consider G as an affine group scheme over $\mathbf{C}$. Then G has an obvious faithful representation W of dimension 1 over $\mathbf{C}$, sending the generator σ to a primitive mth root of unity in $\mathbf{C}^*$. Then G acts freely outside the origin on $W^{\oplus N+1}$, for any natural number N. Therefore, the definition of CH^iBG gives that, for $0 \leq i \leq N$,

$$CH^iBG = CH^i((A^{N+1} - 0)/G).$$

Here $Y := (A^{N+1} - 0)/G$ has an obvious morphism to $(A^{N+1} - 0)/G_m = \mathbf{P}^N$, with fibers $G_m/G \cong A^1_k - 0$. Explicitly, one can check that Y is the total space of the line bundle $O(m)$ over $\mathbf{P}^N$ minus the zero section. Therefore, the Chow ring of Y is the Chow ring of $\mathbf{P}^N$ (namely, $\mathbf{Z}[u]/(u^{N+1})$) modulo the ideal generated by $c_1O(m) = mu$. Letting N go to infinity, it follows that

$$CH^*B\mathbf{Z}/m = \mathbf{Z}[u]/(mu).$$

Example 2.2.4. The Chow ring of $BGL(n)$ over any field k is the polynomial ring

$$CH^*BGL(n) = \mathbf{Z}[c_1, \ldots, c_n]$$

with $|c_i| = i$. These generators are called *Chern classes.*

To explain the name: for any affine group scheme G of finite type over a field k, the Chow ring of BG is isomorphic to the ring of *characteristic classes* for principal G-bundles over smooth k-schemes with values in the Chow ring [7, Theorem 1.3]. Here a *principal G-bundle* is defined in the most general sense: locally trivial in the flat topology. If G is smooth over k, then these are the same as principal G-bundles for the étale topology. For some "special" groups G (such as G_m, $GL(n)$, and $SL(n)$), these are also the same as principal G-bundles for the Zariski topology. A *characteristic class* means an assignment α to every principal G-bundle E over a smooth k-scheme X of an element $\alpha(E)$ in CH^*X, such that α of the pullback of a G-bundle E by any morphism $Y \to X$ is the pullback of $\alpha(E)$ to CH^*Y.

Thus, given the equivalence between principal $GL(n)$-bundles and vector bundles of rank n, the calculation of $CH^*BGL(n)$ describes all characteristic classes of vector bundles with values in the Chow ring. The generators $c_1, \ldots, c_n \in CH^*BGL(n)$ are the usual Chern classes for vector bundles.

In each of these examples (G_m, a finite cyclic group, or $GL(n)$), when the base field is the complex numbers, the Chow ring of BG maps isomorphically to the integral cohomology of BG. That fails for many other groups, as we will see.

Chapter 3

Lecture 3

Today, I will discuss Euler classes in cohomology and in Chow groups. Then I will introduce equivariant Chow groups, following Edidin and Graham. Finally, I will discuss the difference between the Chow ring and integral cohomology for BG, which occurs already for abelian groups G.

3.1 Euler classes

3.1.1 The Gysin homomorphism in topology

Let Y be a real manifold, $X \subset Y$ a closed submanifold (not necessarily compact), $f\colon X \hookrightarrow Y$ the inclusion. Suppose that we are given an orientation on the normal bundle of X in Y, $N_{X/Y}$. Then we can define the "Gysin homomorphism"

$$f_*\colon H^i(X, \mathbf{Z}) \to H^{i+r}(Y, \mathbf{Z}),$$

where $r = \operatorname{codim}(X \subset Y)$. To define this, use the map of pairs

$$(Y, \emptyset) \to (Y, Y - X) \simeq \operatorname{Th}_X N_{X/Y},$$

the Thom space of the normal bundle. So we have a pullback homomorphism

$$H^i(X) \cong H^{i+r}(\operatorname{Th}_X N_{X/Y}) \to H^{i+r}Y,$$

where the first isomorphism is the Thom isomorphism theorem. (This is where the orientation of $N_{X/Y}$ is used.) Geometrically, the map f_* should be easy to visualize: for an element of $H^i(X, \mathbf{Z})$ represented by a codimension-i submanifold S of X, just view S as a submanifold of Y, where it has codimension $i + r$.

There are various other ways to define the Gysin homomorphism. For example, assume that X and Y are both oriented (which gives an orientation of $N_{X/Y}$). Then we can define the Gysin homomorphism as the composition

$$H^iX \cong H^{BM}_{\dim(X)-i}X \xrightarrow[f_*]{} H^{BM}_{\dim(X)-i}Y \cong H^{i+r}Y,$$

using Poincaré duality on X and Y, where Y has dimension $\dim(X) + r$. Here f_* denotes proper pushforward on Borel-Moore homology.

It is natural to ask: what happens if we push forward and then pull back? There is a simple answer, sometime called the *self-intersection formula*. In the situation above,

$$f^*f_*(u) = u\,\chi(N_{X/Y})$$

for any $u \in H^*(X, \mathbf{Z})$. Here $\chi(E)$ is the *Euler class* in $H^r(X, \mathbf{Z})$ of an oriented rank-r real vector bundle E. (A standard reference for the Euler class in topology is Milnor-Stasheff's *Characteristic Classes*.)

3.1.2 The self-intersection formula for Chow groups

Let $f\colon X \hookrightarrow Y$ be a closed embedding of smooth k-schemes. Then, for any $u \in CH^*X$, we have

$$f^*f_*(u) = u\,c_r(N_{X/Y}),$$

where $r = \mathrm{codim}(X \subset Y)$. Thus the top Chern class plays the role of the Euler class here. (That makes sense: for a complex vector bundle E on a topological space, E has a canonical orientation as a real vector bundle, and the Euler class of E as an oriented real bundle is the top Chern class of E in cohomology.)

3.1.3 The Chow ring of an $(A^r - 0)$-bundle

Let E be a vector bundle of rank r on a smooth scheme X over a field k. Using the self-intersection formula plus the localization sequence, we can compute the Chow ring of the total space of E minus the zero-section (isomorphic to X):

$$CH^*(E - X) \cong CH^*(X)/(c_r(E)).$$

We saw this formula for E a line bundle in lecture 2.

Note the difference from what happens in cohomology, for $k = \mathbf{C}$. Namely, in the classical topology, $E - X$ is homotopy equivalent to an S^{2r-1}-bundle over X. By the spectral sequence of the fibration, the cohomology of $E - X$ contains $H^*(X, \mathbf{Z})/(c_r(E))$, but it may be bigger.

3.2 Equivariant Chow groups

3.2.1 Equivariant cohomology

We begin with the definition of equivariant cohomology. (The version here may be called "Borel equivariant cohomology.") Let G be a topological group acting on a topological space X. Let EG be a contractible free G-space. Then the G-equivariant cohomology of X is defined by

$$H^i_G(X, \mathbf{Z}) := H^i((X \times EG)/G, \mathbf{Z}).$$

This is independent of the choice of EG. (In short: we replace X by a homotopy equivalent space on which G acts freely, and then take the quotient space.) The space $(X \times EG)/G$ may be called the *Borel construction* or the *homotopy quotient* $X//G$.

Example 3.2.1. $H^i_G(\text{point}) = H^i BG$, by definition of BG.

Example 3.2.2. If G acts freely on X, then $H^i_G(X) = H^i(X/G)$.

Thus equivariant cohomology puts the problem of computing the cohomology of quotients by free G-actions in a broader context, since we also get invariants of G-actions that are not free. This flexibility is useful for computations.

It is immediate that $H^*_G(X, \mathbf{Z})$ is a graded-commutative ring (since it is the cohomology of a space), and that every G-equivariant continuous map $f\colon X \to Y$ determines a pullback ring homomorphism $f^*\colon H^*_G(Y, \mathbf{Z}) \to H^*_G(X, \mathbf{Z})$. (A map $f\colon X \to Y$ of G-spaces is *G-equivariant* if $f(gx) = gf(x)$ for all $g \in G$, $x \in X$.)

To compute equivariant cohomology, note that we have a fibration

$$X \to (X \times EG)/G \to EG/G = BG.$$

So we have a spectral sequence

$$E_2^{ij} = H^i(BG, H^j X) \Rightarrow H^{i+j}_G X.$$

3.2.2 Equivariant Chow groups

Equivariant Chow groups were defined by Edidin and Graham [1], using my algebro-geometric construction of BG. Namely, let X be a scheme with an action of an affine group scheme G over a field k. Let i be an integer. Let V be any representation of G such that G acts freely on a closed subset $S \subset V$ with $\operatorname{codim}(S \subset V) > \dim(X) - i$. Then we define the ith *equivariant Chow group* by

$$CH^G_i X = CH_{i+\dim(V)-\dim(G)}((X \times (V - S))/G).$$

By the same proof as for CH^*BG, this group is independent of the choice of (V, S).

In these notes, I will only consider equivariant Chow groups for X smooth over k. In that case, we define $CH^i_G X = CH^G_{\dim(X)-i} X$, and $CH^*_G X$ is a commutative graded ring. The definition looks simpler with this numbering:

$$CH^i_G X = CH^i((X \times (V - S))/G)$$

for $\operatorname{codim}(S \subset V) > i$. For an equivariant morphism $f\colon X \to Y$ of smooth G-schemes over k, we have a pullback ring homomorphism

$$f^*\colon CH^*_G Y \to CH^*_G X.$$

Example 3.2.3. $CH^i_G(\operatorname{Spec} k) = CH^*BG$, by definition of BG.

Example 3.2.4. If G acts freely on X, then $CH^i_G X \cong CH^i(X/G)$. (To check this, use homotopy invariance for Chow groups.)

We have a "fibration" in algebraic geometry (as in topology),

$$X \to X//G \to BG.$$

(More concretely, one can consider the analogous fibration of finite-dimensional approximations, $X \to (X \times (V - S))/G \to (V - S)/G$.) Unfortunately, we do not have a spectral sequence for such a fibration in terms of Chow groups (or even motivic cohomology). One difficulty is that this fibration will typically be étale-locally trivial but not Zariski-locally trivial, and "étale descent" does not hold for Chow groups, unless we tensor with $\mathbf{Q}$. Nonetheless, it is useful to think about this fibration when you want to compute equivariant Chow groups. For example, this makes it clear that $CH^*_G X$ is a module over CH^*BG.

3.2.3 Solution to problem 1 from problem sheet 2

Problem: Let G be the multiplicative group G_m over a field k. Let G act on the affine plane A^2 over k by

$$t(x,y) = (tx, t^{-1}y).$$

Compute the G-equivariant Chow ring $CH^*_G(A^2 - 0)$.

(Side question: What is the geometric quotient $(A^2 - 0)/G$? This is not a separated scheme, so it's outside the usual setting where I defined Chow groups.)

Solution: Use the *localization sequence* for equivariant Chow groups:

$$CH^i_G(\{0\}) \to CH^{i+2}_G(A^2) \to CH^{i+2}_G(A^2 - 0) \to 0.$$

(This is immediate from the non-equivariant localization sequence, since equivariant Chow groups (in a given degree) are defined as the Chow groups of an associated scheme.) Here $CH^i_G(\{0\}) = CH^iBG$, and likewise $CH^{i+2}_G A^2 \cong CH^{i+2}BG$. (Indeed, by the fibration above, the G-equivariant Chow ring of A^2 is the Chow ring of the total space of a rank-2 vector bundle over BG, and this is isomorphic to CH^*BG by homotopy invariance of Chow groups.)

So what is the homomorphism $CH^iBG \to CH^{i+2}BG$? From the fibration above, this is the pushforward associated to the inclusion of BG into the vector bundle over BG associated to the representation $E = A^2$ of G we started with. By the self-intersection formula, this homomorphism is multiplication by the top Chern class of E, $c_2(E) \in CH^2BG$.

By our definition of E, E is the direct sum of two 1-dimensional representations of $G = G_m$, $t \mapsto t$ and $t \mapsto t^{-1}$. If we call the first representation L, the second one is the dual representation L^*. These 1-dimensional representations determine line bundles on the approximating spaces $\mathbf{P}^N$ to BG_m; explicitly, we can say that $L = O(1)$ and $L^* = O(-1)$ on $\mathbf{P}^N$. Let $u = c_1O(1)$, which is a generator of $CH^1BG_m = \mathbf{Z}$. Then

$$\begin{aligned} c_2(E) &= c_2(L \oplus L^*) \\ &= c_1(L)c_1(L^*) \\ &= u(-u) \\ &= -u^2. \end{aligned}$$

Therefore, the localization sequence above gives that

$$\begin{aligned} CH^*_G(A^2 - 0) &\cong CH^*BG/(c_2(E)) \\ &\cong \mathbf{Z}[u]/(-u^2) \\ &\cong \mathbf{Z}[u]/(u^2). \end{aligned}$$

This happens to be isomorphic to the Chow ring of $\mathbf{P}^1$, which is a *different* quotient scheme $(A^2 - 0)/G_m$ (namely, with G_m acting by $t(x,y) = (tx, ty)$).

To answer the second part of the problem: the quotient $(A^2 - 0)/G_m$ (with G_m acting by $t(x,y) = (tx, t^{-1}y)$) is outside the usual setting where I defined Chow groups, since it is non-separated. Namely, xy is a G_m-invariant function on A^2, which gives a morphism $(A^2 - 0)/G_m \to A^1$. But this is not an isomorphism, because the fiber over 0 consists of two orbits, the x-axis minus the origin and the

y-axis minus the origin. As a result, $(A^2 - 0)/G_m$ is the *line with two origins*, the union of two copies of A^1, with the two open subsets $A^1 - 0$ identified by the identity map. You can convince yourself that, over $\mathbf{C}$, the line with two origins should be viewed as having the homotopy type of the 2-sphere, just like $\mathbf{CP}^1$.

Edidin and Graham show that the usual definition of Chow groups actually works, with the usual properties, for all algebraic spaces of finite type over a field [1, section 6.1]. This includes all schemes of finite type over a field, separated or not.

3.3 The Chow ring of an elementary abelian group

We describe here the Chow ring of $B(\mathbf{Z}/p)^n$ over the complex numbers. This is a simple case where the Chow ring differs from integral cohomology (for $n \geq 2$).

Let p be a prime number. We have seen that the Chow ring of $B\mathbf{Z}/p$ over $\mathbf{C}$ is $\mathbf{Z}[y]/(py)$, where $y \in CH^1 B\mathbf{Z}/p$ is the first Chern class of a faithful 1-dimensional representation, $\mathbf{Z}/p \hookrightarrow \mathbf{C}^*$. The calculation used that we can take the approximations to $B\mathbf{Z}/p$ to be an $(A^1 - 0)$-bundle over $\mathbf{P}^N$ for N large, namely the complement of the zero section in the line bundle $O(p)$ over $\mathbf{P}^N$.

As a result, for a positive integer n, we can approximate $B(\mathbf{Z}/p)^n$ by the product of n copies of the spaces above. That is, we have an $(A^1 - 0)^n$-bundle over $(\mathbf{P}^N)^n$. We know the Chow ring of $(\mathbf{P}^N)^n$ by the projective bundle theorem, and we can apply our description of the Chow ring of an $(A^1 - 0)$-bundle n times. The result is:

$$CH^* B(\mathbf{Z}/p)^n = \mathbf{Z}[y_1, \ldots, y_n]/(py_1, \ldots, py_n).$$

This maps isomorphically to $H^*(B(\mathbf{Z}/p)^n, \mathbf{Z})$ for $n = 1$, but not for larger n. You could say that this happens because of the Tor term in the Künneth formula for integral cohomology. In particular, for $n \geq 2$, $B(\mathbf{Z}/p)^n$ has some cohomology in odd degrees, which certainly cannot come from the Chow ring (since $CH^i BG$ maps to $H^{2i}(BG, \mathbf{Z})$). For $n \geq 3$, the Chow ring does not even map onto the even-degree integral cohomology.

It is easier to describe the difference between the Chow ring and cohomology with mod p coefficients. (The Chow ring of a smooth k-scheme X with coefficients in a commutative ring R just means $CH^*(X) \otimes_{\mathbf{Z}} R$.) For example, let p be an odd prime number. Then the Chow ring of $B(\mathbf{Z}/p)^n$ modulo p is the polynomial ring

$$CH^*(B(\mathbf{Z}/p)^n)/p = \mathbf{F}_p[y_1, \ldots, y_n],$$

whereas the mod p cohomology ring is a free graded-commutative algebra:

$$H^* B(\mathbf{Z}/p)^n, \mathbf{F}_p) = \mathbf{F}_p\langle x_1, \ldots, x_n, y_1, \ldots, y_n\rangle$$

with $|x_i| = 1$ and $|y_i| = 2$. (The elements x_i generate an exterior algebra, by graded-commutativity.) The elements y_i in the Chow ring map to the polynomial generators y_i in cohomology. Of course the Chow ring cannot map to odd-degree elements such as the x_i's, but also even-degree elements such as x_1x_2 are not in the image of the Chow ring.

For $p = 2$, there are some differences in the story: $H^*(B(\mathbf{Z}/2)^n, \mathbf{F}_2)$ is the polynomial ring on generators $x_1, \ldots, x_n$ of degree 1, and $CH^*(B(\mathbf{Z}/2)^n)/2 =$

$\mathbf{F}_2[y_1, \ldots, y_n]$ maps by $y_i \mapsto x_i^2$. Again, the Chow ring misses a lot of the cohomology.

One way to describe what happens for these groups G is that the image of CH^*BG in $H^*(BG, \mathbf{Z})$ is exactly the subring generated by Chern classes of complex representations of G. That happens for some other groups G, which makes sense because Chern classes of representations live in the Chow ring. But it's not true for all finite groups. In some examples such as the symmetric groups $G = S_n$, CH^*BG is generated at least by *transfers* of Euler classes of representations of subgroups of G [8, Theorem 2.22]. But even that fails in general.

The next lecture will show a striking failure of this kind of statement: at least over some extension fields k of $\mathbf{C}$, there are finite groups G for which the abelian group CH^iBG_k is not even finitely generated.

Chapter 4

Lecture 4

In this final lecture, I will explain some negative results about finite generation for the Chow groups of classifying spaces. We use the technique of "decomposition of the diagonal" for studying algebraic cycles, used most famously by Bloch, Srinivas, and Voisin.

4.1 Positive results on finite generation

Theorem 4.1.1. *For an affine group scheme G over a field k with a faithful representation V of dimension n, the ring CH^*BG is generated by elements of degree at most $n(n-1)/2$ if $n \geq 3$, or of degree at most n if $n \leq 2$.*

The proof uses the fibration $GL(n)/G \to BG \to BGL(n)$ [8, Theorems 5.1 and 5.2]. See problem sheet 3 for a related result.

Therefore, CH^*BG is a finitely generated $\mathbf{Z}$-algebra if and only if CH^iBG is a finitely generated abelian group for each i. Does the latter statement hold? In many examples, it does hold. Moreover, in examples one often finds that the Chow ring CH^*BG_k is the same for all sufficiently large fields k.

Example 4.1.2. (R. E. Field [3]) For every field k of characteristic not 2, and any positive integer m, let q be the "split" quadratic form $x_1x_2 + x_3x_4 + \cdots + x_{2m-1}x_{2m}$ over k. Then the group scheme $SO(2m) := SO(q)$ over k has

$$CH^*BSO(2m) = \mathbf{Z}[c_2, c_3, \ldots, c_{2m}, y_m]/(2c_{\text{odd}}, y_m c_{\text{odd}}, y_m^2 - (-1)^m 2^{2m-2} c_{2m}).$$

For $k = \mathbf{C}$, y_m maps to 2^{m-1} times the Euler class in $H^{2m}(BSO(2m), \mathbf{Z})$.

It turns out that the abelian groups CH^iBG_k are *not* finitely generated in general, even for finite groups G. However, our counterexamples have base field k a big, "artificial" field. It is completely open whether the groups CH^iBG_k are finitely generated for k algebraically closed.

The main tool for our counterexamples is the following theorem, which relates several good properties for an algebraic variety. The theorem is from my paper [9, Theorem 2.1], extending earlier work by Bloch, Merkurjev, Jannsen, and others.

Theorem 4.1.3. *Let X be a smooth proper variety over a field k. The following are equivalent.*

1. *For every field F containing k, the pullback homomorphism $CH_0(X) \to CH_0(X_F)$ is surjective.*

2. *For every field F containing k, the degree homomorphism* $\deg\colon CH_0(X_F) \to \mathbf{Z}$ *is an isomorphism.*

3. *The birational motive of X (in the sense of Kahn-Sujatha) is isomorphic to the birational motive of a point.*

4. *For every cycle module M (in the sense of Rost), the homomorphism $M(k) \to M(k(X))_{nr}$ is an isomorphism. (That is, X has trivial unramified cohomology in the most general sense.)*

5. *There is a nonempty open subset $U \subset X$ such that $CH_i(U_F) = 0$ for every field F containing k and every $i < \dim(X)$.*

6. *There is a nonempty open subset $U \subset X$ such that $CH_i U \to CH_i(U_F)$ is surjective for every field F containing k and every i.*

Let me discuss these properties. Start with property (1), that the Chow group of zero-cycles of X does not increase when you increase the base field. (It is *not* enough to consider finite extensions of the base field here. Property (1) is nontrivial even when k is algebraically closed.) For example, property (1) is true for projective space $\mathbf{P}^n$ over k, since $CH_0(\mathbf{P}^n_F) = \mathbf{Z}$ for every extension field F of k. On the other hand, for an elliptic curve E over a field k, we have

$$CH_0(E) \cong \mathbf{Z} \oplus E(k),$$

using the group structure on the set $E(k)$ of k-rational points. When you increase the base field, you will typically increase the set of rational points of E; so property (1) fails for an elliptic curve. (More generally, it fails for a smooth projective curve of any genus at least 1.) So property (1) picks out a class of varieties that are "like" projective space and not like a curve of higher genus, in terms of the Chow group of zero-cycles.

The equivalence of (1) and (2) is already surprising. This says that if the Chow group of zero-cycles does not increase under field extensions, then it is isomorphic to $\mathbf{Z}$, and it remains $\mathbf{Z}$ over every field extension. You could say that $\mathbf{Z}$ is the only "natural" value for the Chow group of zero-cycles on a smooth proper variety.

Two varieties are said to be *birational* if they have nonempty open subsets that are isomorphic. (Remember that we are using the Zariski topology; so a nonempty open subset of a variety is pretty big, the complement of a lower-dimensional subset.) Equivalently, two varieties over k are birational if their function fields are isomorphic (as fields containing k). The Chow group of zero-cycles is known to be birationally invariant (by Colliot-Thélène, Sansuc, and Fulton), and so properties (1) and (2) are birational invariants of X.

Kahn and Sujatha's category of birational motives is easy to define. The objects are the smooth proper k-varieties, and the set of morphisms from the motive of X to the motive of Y is the abelian group $CH_0(Y_{k(X)})$. (Composition of morphisms is given by composing correspondences.) It is straightforward to relate (1) and (2) to (3).

Property (4) relates properties (1)-(3) to a quite different class of birational invariants, unramified cohomology. Rather than discussing the most general notion of unramified cohomology as in (4), let me describe a special case which is all we will need for the application to BG. Let k be a field, n a positive integer invertible in k, and $i \geq 0$. Then one type of unramified cohomology (for a smooth proper variety X over k) is the group $H^0_{\text{Zar}}(X, \mathcal{H}^i_{\text{ét},\mathbf{Z}/n(i)})$. (In particular, this group is a birational invariant of X.) By definition, this is the group of global sections of the sheaf associated to the presheaf $U \mapsto H^i_{\text{ét}}(U, \mathbf{Z}/n(i))$, where $\mathbf{Z}/n(i)$ means the étale sheaf $\mu_n^{\otimes i}$. By the Bloch-Kato conjecture (proved by Voevodsky and Rost), we can also describe this group as the global sections of the Milnor K-theory sheaf modulo n, $H^0(X, K_i^M/n)$.

Another description of this group is that it is the subgroup of elements u in the Galois cohomology group $H^i(k(X), \mathbf{Z}/n(i))$ such that the residue of u in $H^{i-1}(k(D), \mathbf{Z}/n(i-1))$ is zero for every codimension-1 subvariety D in X. (That is, u is "unramified" along every codimension-1 subvariety.) Note that $H^i(k(X), \mathbf{Z}/n(i))$ is trivially a birational invariant of X, since it only depends on the field $k(X)$; but it is too big to be useful. Unramified cohomology is a subgroup which is still birationally invariant, but small enough to be useful.

Example 4.1.4.

$$H^0(X, \mathcal{H}^1_{\mathbf{Z}/n(1)}) \cong H^1(X, \mathbf{Z}/n(1)).$$

Thus $H^1(X)$ is birationally invariant for smooth proper varieties X; this would not be true for higher-degree cohomology. For example, this implies that two smooth projective curves of different genera cannot be birational. There are easier ways to prove that, though, and this invariant is not very powerful.

Example 4.1.5.

$$H^0(X, \mathcal{H}^2_{\mathbf{Z}/n(1)}) \cong \mathrm{Br}(X)[n],$$

the n-torsion subgroup of the Brauer group. Thus the Brauer group is birationally invariant for smooth proper varieties. This is a much more subtle invariant, as we will see. In particular, the Brauer group can be nontrivial for a unirational variety over an algebraically closed field (thereby showing that such a variety is not rational). By definition, a variety X is *unirational* over a field k if there is a dominant rational map from some projective space to X.

Proof. (Theorem 4.1.3) I will only prove the equivalence of (1) and (2). This is a classic application of the technique of "decomposition of the diagonal", which is also used for the other equivalences. The equivalence of (1)-(3) and (4) was shown by Merkurjev [5], and the new aspect of my paper was the equivalence of (1)-(4) with (5) and (6).

So let us prove that (1) implies (2). (It is clear that (2) implies (1).) Let X be a smooth proper variety over a field k such that $CH_0(X) \to CH_0(X_F)$ is surjective for every field F over k. Let n be the dimension of X. The first key idea is to apply the assumption to the field $F = k(X)$, the function field of X.

How can we describe $CH_0(X_{k(X)})$, or more generally $CH_0(Y_{k(X)})$ for another variety Y over k? We know that $CH_0(Y_{k(X)})$ is generated by the 0-dimensional subvarieties (i.e, the closed points) in the scheme $Y_{k(X)}$, but what are they? The point is to think of $Y_{k(X)}$ as the *generic fiber* of the projection $X \times_k Y \to X$, via the pullback diagram:

$$\begin{array}{ccc} Y_{k(X)} & \longrightarrow & X \times_k Y \\ \downarrow & & \downarrow \\ \operatorname{Spec} k(X) & \longrightarrow & X. \end{array}$$

Each 0-dimensional subvariety of $Y_{k(X)}$ is the generic fiber (over X) of an n-dimensional subvariety of $X \times Y$ that dominates X (i.e., whose image is dense in X). (And this gives a one-to-one correspondence between these two classes of subvarieties.)

Returning to the case $Y = X$: the *diagonal* $\Delta_X \subset X \times X$ is an n-dimensional subvariety that dominates X by the first projection $\pi_1 \colon X \times X \to X$. Therefore, the generic fiber of Δ_X over the first copy of X is a closed point in $X_{k(X)}$, call it $[\Delta_X]$. In fact, it is a closed point of degree 1 (a $k(X)$-rational point), because $\Delta_X \subset X \times X$ has degree 1 over the first copy of X.

We are given that $CH_0(X) \to CH_0(X_{k(X)})$ is surjective. So there is a 0-cycle α on X (over the base field k) such that

$$[\Delta_X] = \alpha$$

in $CH_0(X_{k(X)})$. (Since the 0-cycle $[\Delta_X]$ has degree 1, so does α.) Equivalently, the n-dimensional cycles Δ_X and $X \times \alpha$ in $X \times X$ become rationally equivalent over the generic fiber of the first projection π_1. In fact, we can reformulate this statement in terms of the Chow groups of $X \times X$: $CH_0(X_{k(X)})$ is the quotient of $CH_n(X \times X)$ by the subgroup generated by all n-dimensional subvarieties whose image under π_1 is not dense in X. (That is, we kill all subvarieties whose generic fiber in $X_{k(X)}$ is empty.)

Therefore, the equality above implies that

$$\Delta_X = X \times \alpha + B$$

in $CH_n(X \times X)$, for some n-dimensional cycle B on $X \times X$ whose image in the first copy of X is contained in some closed subset $S \subsetneq X$. This is called a *decomposition of the diagonal* for X.

We now use the idea of correspondences. Namely, for smooth proper varieties X and Y over a field k, an element u of $CH_*(X \times Y)$ (called a *correspondence* from X to Y) determines a homomorphism from $CH_*(X)$ to $CH_*(Y)$. (Namely: given a cycle β on X, pull it back to $X \times Y$, intersect with u, and then push forward to Y. Call this $u_*\beta \in CH_*Y$.) The idea is extremely flexible: a correspondence also induces a homomorphism from $CH_*(Y)$ to $CH_*(X)$, from $H_*(X, \mathbf{Z})$ to $H_*(Y, \mathbf{Z})$ (if $k = \mathbf{C}$), and so on. These various ways of using correspondences give various ways to apply the decomposition of the diagonal above, and this idea is used to prove the equivalences of (1) to (6) in Theorem 4.1.3. For example, we constructed a decomposition of the diagonal using assumption (1) on the Chow group of zero cycles, but we can then apply that decomposition to get information about Chow groups in other dimensions, as in (5) and (6).

For now, we focus on proving that (1) implies (2). We have shown that (1) implies a decomposition of the diagonal as above. As a correspondence, the diagonal $\Delta_X \subset X \times X$ induces the identity from $CH_i(X)$ to $CH_i(X)$ for all i. (Think of this operation as going from cycles on the first copy of X to the second.) Therefore, for every zero-cycle β in $CH_0(X)$,

$$\begin{aligned}\beta &= \Delta_*\beta \\ &= (X\times\alpha)_*\beta + B_*\beta \\ &= (X\times\alpha)_*\beta.\end{aligned}$$

Here we use a version of Chow's moving lemma to get that every 0-cycle $\beta \in CH_0(X)$ is rationally equivalent to a cycle disjoint from the closed subset $S \subsetneq X$, above. Since the cycle B is supported in $S \times X$, it is then clear (by definition of correspondences, above) that $B_*\beta = 0$. Furthermore, the 0-cycle $(X \times \alpha)_*\beta$ is clearly supported on the support of α (draw a picture), and with more care you can see that it is exactly $\deg(\beta)\alpha$.

Thus we have shown that CH_0X is generated by α. Since α has degree 1, it follows that the degree homomorphism $\deg\colon CH_0(X) \to \mathbf{Z}$ is an isomorphism. For statement (2), we want to prove the same statement over every extension field F of k. But this is easy: just observe that our decomposition of the diagonal

$$\Delta_X = X \times \alpha + B$$

in $CH_n(X \times_k X)$ implies the same type of decomposition in $CH_n(X_F \times_F X_F)$ for every field F over k. Then the same argument implies that $\deg\colon CH_0(X_F) \to \mathbf{Z}$ is an isomorphism for every field F over k. We have shown that (1) implies (2). The other parts of Theorem 4.1.3 use the same kind of arguments. QED

4.2 Failure of finite generation for $CH^i BG$

In my book, I computed the Chow rings of all p-groups G of order at most p^4 with $p = 2$ or 3, and for 13 of the 15 groups of order p^4 with p a prime at least 5 [8, Chapter 13]. It turns out that these are finitely generated $\mathbf{Z}$-algebras, and (in each case) the Chow ring is the same over all fields of characteristic zero that contain the $|G|$ roots of unity.

However, by Saltman, Bogomolov, Hoshi, Kang, and Kunyavskii, there are groups G of order p^5 for every odd prime number p, and groups of order $2^6 = 64$, with nontrivial Brauer group. That is, for any faithful representation V over a field k of characteristic zero, with G acting freely on an open set $U \subset V$, if we choose a smooth compactification $U/G \subset X$, then $\mathrm{Br}(k) \to \mathrm{Br}(X)$ is not an isomorphism. (So X is not stably rational, although it is obviously unirational.) The following Corollary says that the Chow rings of these groups need not be finitely generated, over some fields k.

Corollary 4.2.1. *([9, Corollaries 3.1 and 3.2].) Let G be a finite group, k a field of characteristic zero, p a prime number. Suppose that V has a faithful representation V, with G acting freely on an open set $U \subset V$, such that a smooth compactification of U/G has nontrivial unramified cohomology (with coefficients in some $\mathbf{F}_p$-cycle module). Then there is an $i \geq 0$ and a field F over k such that*

$$CH^i(BG_k)/p \to CH^i(BG_F)/p$$

is not surjective.

Moreover, there are fields F over k for which the cardinality of $CH^i(BG_F)/p$ is as big as we like. In particular, $CH^i(BG_F)/p$ can be infinite, and so the abelian group $CH^i(BG_F)$ need not be finitely generated.

Proof. Let V be a faithful representation of G over k, and let $S \subsetneq V$ be a G-invariant closed subset such that G acts freely on $V - S$. We are given that some smooth compactification X of $(V-S)/G$ has nontrivial unramified cohomology. By Theorem 4.1.3, that implies something about the Chow groups of the open subset $(V-S)/G$ of X. Namely, there is some $i \geq 0$ and some field F over k such that

$$CH^i((V-S)/G) \to CH^i((V_F - S_F)/G)$$

is not surjective. More precisely, there is a version of Theorem 4.1.3 with $\mathbf{F}_p$ coefficients. Since X has nontrivial unramified cohomology with coefficients in some $\mathbf{F}_p$-cycle module, it follows that

$$CH^i((V-S)/G)/p \to CH^i((V_F - S_F)/G)/p$$

is not surjective.

We want to deduce the corresponding statement about BG; this is not immediate, because we do not know whether i is greater than $\operatorname{codim}(S \subset V)$. Fortunately, we can use the localization sequence for equivariant Chow groups to see that

$$CH^i BG = CH^i_G V \to CH^i_G(V-S) = CH^i((V-S)/G)$$

is surjective, and likewise $CH^i(BG_F) \to CH^i((V_F - S_F)/G)$ is surjective.

Consider the commutative square

$$\begin{array}{ccc} CH^i(BG)/p & \longrightarrow & CH^i((V-S)/G)/p \\ \downarrow & & \downarrow \\ CH^i(BG_F)/p & \longrightarrow & CH^i((V_F - S_F)/G)/p. \end{array}$$

Since the horizontal maps are surjective and the right vertical map is not surjective, the left vertical map must also not be surjective. That is, $CH^i(BG)/p \to CH^i(BG_F)/p$ is not surjective, as we want. We omit the details of showing that $CH^i(BG_F)/p$ can have arbitrarily large cardinality. QED

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