PCMI notes 1: Chow groups

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1 Introduction

One of the main methods of complex algebraic geometry is to think of a complex algebraic variety as a complex manifold and in particular a topological space, and look at its ordinary cohomology. Roughly speaking, that means that we think about all possible real submanifolds of the given complex variety. But an algebraic variety is more than just a topological space. In particular, it is important to ask which cohomology classes can be represented by complex algebraic subvarieties. This is the subject of the theory of algebraic cycles and in particular the Hodge conjecture, one of the famous seven "million-dollar problems" in mathematics.

The key concept in the course is the notion of the Chow groups of an algebraic variety over any field. These are groups with some of the same formal properties as homology or cohomology groups, but they are built directly from the algebraic subvarieties of a given variety. A big difference from cohomology is that Chow groups are extremely hard to compute in general. In fact, computing Chow groups for arbitrary varieties would amount to solving the Hodge conjecture and many other conjectures of algebraic geometry and number theory (such as the Birch-Swinnerton-Dyer conjecture, another million-dollar problem). Nonetheless, in this course we will see how to compute the Chow groups at least for some varieties.

Chow groups can be seen as a first step in bringing methods from homotopy theory into algebraic geometry. Some of the later steps in this direction are algebraic K-theory, motivic cohomology, and algebraic cobordism.

Problem set 1 is related to this first set of notes.

Here are some of the relevant books. For general algebraic geometry, a standard reference is R. Hartshorne, *Algebraic Geometry* (Springer). Many other books cover similar material, such as Ravi Vakil's *The Rising Sea*, free on the web. For example, in Hartshorne, section II.6 on divisors is an excellent geometric treatment, which is the most direct background needed for Chow groups. Chapter IV on curves shows how to use divisors and line bundles to solve geometric problems. Also, Appendix A is a good short summary of the Chow ring, as covered in these notes.

W. Fulton's *Intersection Theory* (Springer) is the basic reference on Chow groups. It is a densely written book, so it can be hard to read straight through. But the basic Chapter 1 on Chow groups is very readable, and you can skip around through the rest. The book has a massive number of useful examples. Finally, EIsenbud-Harris's *3264 and All That* (Cambridge), is a more elementary introduction to Chow groups and how to compute with them. It is in the PCMI library here.

Before defining Chow groups, I will discuss divisors and line bundles, which is the foundation of the theory of Chow groups. Some of this should be familiar, but it's important to understand how these things work on singular varieties, which may be new to you.

2 A few words on schemes

In this course a "scheme" will usually mean a separated scheme of finite type over a field k, while a "variety" means a reduced irreducible scheme. However, very little will be lost if you consider only quasi-projective schemes over k. The difference between schemes and varieties is: first, a scheme can have several irreducible components, and second, a scheme may be non-reduced, meaning that the ring of regular functions on a scheme is allowed to have nilpotent elements. There is a common sloppy use of "variety" to mean any reduced scheme (meaning that it is allowed to have several irreducible components); I will try to avoid that usage, but I will probably fail on occasion.

The difference between schemes and varieties comes up naturally when we want to describe "multiplicities". For example, let $f(x)$ be a polynomial in one variable over an algebraically closed field k, and consider the closed subscheme X of the affine line A^1 defined by $f = 0$. Then the ring of regular functions $O(X)$ is $k[x]/(f(x))$. This ring $O(X)$ is reduced if and only if f has d distinct roots; in that case, $O(X)$ is a product of d copies of k, where d is the degree of f. If f has repeated roots, then the scheme X is not reduced. Explicitly, if f factors as

$$f = (x - a_1)^{r_1} \cdots (x - a_m)^{r_m},$$

then the ring of functions $O(X) = k[x]/(f(x))$ is isomorphic to the product ring

$$k[x]/(x - a_1)^{r_1} \times \cdots \times k[x]/(x - a_m)^{r_m}.$$

Thus the ring $O(X)$ is a product of rings isomorphic to $k[x]/(x^r)$, for various r's. This means that the subscheme $X = \{f = 0\}$ of A^1 remembers not only the set of points where f is zero, but also the number of times $x - a$ appears as a factor of f. This has the convenient result that the 0-dimensional subscheme $X = \{f = 0\}$ of the affine line always has degree d, even though the set of roots of f may have order less than d. (For a 0-dimensional scheme X over k, the ring $O(X)$ is a finite-dimensional k-vector space, and we define the degree of X over k to be the dimension of $O(X)$ as a k-vector space.)

To emphasize that there is nothing mysterious about schemes, let me note that the closed subschemes X of affine space A^n over a field k are in one-to-one correspondence with the ideals I in the polynomial ring $k[x_1, \ldots, x_n]$. (This is just as the closed subvarieties of A^n are in one-to-one correspondence with the prime ideals in $k[x_1, \ldots, x_n]$.) The ring of regular functions on the affine scheme $X \subset A^n$ is simply the quotient ring $k[x_1, \ldots, x_n]/I$, where I is the corresponding ideal. We write $X = \{f_1 = 0, \ldots, f_r = 0\} \subset A^n$ where $f_1, \ldots, f_r$ are some generators of the ideal I. (Thus you can reconstruct the defining ideal of a closed subscheme X of A^n as the kernel of the restriction map $O(A^n) \to O(X)$.)

It is also worth saying a word about closed subschemes of the affine line over a field k which is not algebraically closed. For simplicity, let $f(x)$ be a polynomial over k with no multiple roots (meaning that f and f' have no common factors).

Then X is a reduced subscheme of degree d, the degree of f. It is the disjoint union of several 0-dimensional varieties, corresponding to the irreducible factors of f. A 0-dimensional variety over k is automatically affine, so it has the form $\mathrm{Spec}(E)$ for some integral domain E which is a finite-dimensional k-vector space; those properties imply that E is a field, as you can check. Thus $X = \{f = 0\}$ is the disjoint union of several 0-dimensional varieties $\mathrm{Spec}(E_1), \ldots, \mathrm{Spec}(E_m)$ where E_i are finite field extensions of k whose degrees add up to d. Explicitly, if f is the product of distinct irreducible factors f_i (as we assumed), then the fields E_i are defined by $k[x]/(f_i(x))$.

3 Divisors and line bundles on smooth varieties

References: Hartshorne, "Divisors", II.6; Fulton, appendices A.1-A.3. Hartshorne's section has a lot of good examples.

Line bundles are central to algebraic geometry, for example because all morphisms from a variety to projective space can be described using suitable line bundles. So it is important to study the group of all line bundles on a given scheme. Explicitly, the *Picard group* $\mathrm{Pic}(X)$ of a scheme X means the abelian group of isomorphism classes of line bundles on X. The group operation is defined by the tensor product of line bundles, $L \otimes M$. This is clearly commutative and associative, and the inverse of a line bundle L is the dual line bundle $L^* = \mathrm{Hom}(L, O_X)$.

For smooth schemes X, there is a neat correspondence between line bundles on X and codimension-one subvarieties of X. Precisely, we define an *irreducible divisor* D on X to be a codimension-one closed subvariety of X, meaning that D has dimension one less than that of X. (Remember that varieties are irreducible by definition.) We define the abelian group of *divisors* (or *Weil divisors*) on X to be the free abelian group on the set of irreducible divisors. Thus a divisor is a finite sum $\sum a_i D_i$ where a_i are integers and D_i are irreducible divisors.

The key algebraic fact about divisors on smooth varieties is that every irreducible divisor is locally defined by one equation (as a scheme). More generally, an integral domain is called *factorial* if every codimension-one prime ideal is principal, which is the same property stated in algebraic terms, and we need the theorem from commutative algebra that every regular local ring is factorial.

For a rational function f on X and an irreducible divisor D on X, we can define the *order* of f along D, $\mathrm{ord}_D(f)$, which is an integer. This is positive if f vanishes along D and it is negative if f has a pole along D. For X smooth over a perfect field k, there are very simple ways to define the order of f along D. Since k is perfect, a dense open subset of D is smooth. Passing from k to its algebraic closure, one can pick a k-point in the smooth locus of D. There the completed local ring of X will be isomorphic to the power series ring $k[[x_1, \ldots, x_n]]$, and we can choose these coordinates in such a way that the smooth divisor D is given by $\{x_1 = 0\}$. Then, for a regular function f in a neighborhood of this point which is not identically zero, we define the order of f along D to be the largest power of x_1 which divides the power-series expansion of f. One checks immediately that $\mathrm{ord}_D(fg) = \mathrm{ord}_D(f) + \mathrm{ord}_D(g)$, and so the order extends to a function from the quotient field of the local ring at this point to the integers (with $\mathrm{ord}_D(0)$ not defined). This quotient field contains the field of rational functions on X, so we have defined the order along D of any (nonzero) rational function on X.

We can rephrase the definition of the order in a more natural way (for example, without assuming k is perfect or choosing a point on D) using some algebra. We introduce some terminology: the *local ring* of a scheme X at the generic point of a closed subvariety Z is the direct limit of the rings of regular functions on open subsets U of X that contain at least one point of Z. For X a variety, the local ring at Z can also be described as the ring of rational functions on X which are regular at some point of Z (and therefore on a dense open subset of Z, because Z is irreducible). We use the algebraic theorem that the local ring of a smooth variety at a codimension-r subvariety is a regular local ring of dimension r. Therefore the local ring of X at an irreducible divisor D is a regular local ring of dimension 1, which is called a *discrete valuation ring.*

A discrete valuation ring R has an unique maximal ideal m, which is generated by one element, $m = (g)$; all the powers of g are nonzero; and the intersection of the ideals $m^n = (g^n)$ is zero. (In our geometric setup, g is a regular function on some open subset of X that meets D such that $D = \{g = 0\}$ in this subset.) It follows that we can define the *order* of an nonzero element f in a discrete valuation ring R to be the largest $n \geq 0$ such that f is a multiple of g^n; moreover, from what we have said, f is then automatically g^n times a unit in R. It follows that this is indeed a valuation on R, meaning in particular that $\text{ord}(fg) = \text{ord}(f) + \text{ord}(g)$ for $f, g \in R$. Finally, R is an integral domain, and it follows that the valuation on R extends to the quotient field F of R by defining $\text{ord}(f/g) = \text{ord}(f) - \text{ord}(g) \in \mathbf{Z}$. Again, $\text{ord}_R(0)$ is not defined.

We can then define the divisor (f) or $\text{div}(f)$ of a nonzero rational function f on a smooth scheme X to be the sum

$$\sum_D \text{ord}_D(f) D$$

where the sum runs over all irreducible divisors D. The sum is always finite. Clearly $(fg) = (f) + (g)$, and so the divisors of rational functions form a subgroup of the group $\text{Div}(X)$ of all divisors. We say that two divisors are *linearly equivalent* if their difference can be written as the divisor of some rational function. The *divisor class group* $\text{Cl}(X)$ is the group of linear equivalence classes of divisors. That is, it is the quotient group $\text{Div}(X)/\{(f) : f \in k(X)^*\}$.

Theorem 3.1 *For any smooth scheme (or more generally, any factorial scheme) X, there is a canonical isomorphism, called the first Chern class, from the Picard group of line bundles to the divisor class group:*

$$c_1 : Pic(X) \to Cl(X).$$

Proof. First, under our assumptions, X is normal, so it is the disjoint union of its irreducible components, and we can just assume that X is irreducible.

We first define the homomorphism $c_1 : \text{Pic}(X) \to \text{Cl}(X)$. By definition, any line bundle L is trivial on some open subsets that cover X. Since X is irreducible, it follows that L is trivial on a dense open subset of X. So choose a trivialization of L on U, which we can view as a section s of L over U which is everywhere nonzero. Then we can view s as a rational section of U. (One can define the group of rational sections of L as the direct limit of the groups of sections of L on nonempty open subsets of X; this is a 1-dimensional vector space over the field of rational functions on X.)

We can define the order of vanishing of a rational section s of a line bundle L along an irreducible divisor D by choosing a nonzero section t of L on some open set that meets D, and defining $\mathrm{ord}_D(s) = \mathrm{ord}_D(s/t)$, where s/t is a rational function on X. This does not depend on the choice of t because the ratio of any two trivializations t_1, t_2 is a unit on some open set that meets D, so has order zero along D. Using that, we can define the divisor (s) of a rational section s of a line bundle L as

$$(s) = \sum_D \mathrm{ord}_D(s) D,$$

just as in the case of the trivial line bundle. Finally, we define the *first Chern class* of a line bundle L to be the divisor of any nonzero rational section on L:

$$c_1(L) = (s) \in \mathrm{Cl}(X).$$

Different rational sections s_1, s_2 of L will have different divisors of zeros and poles, but the resulting divisors will be linearly equivalent because

$$(s_2) = (s_1) + (s_2/s_1)$$

where s_2/s_1 is a rational function. Thus we have a well-defined functon $c_1 : \mathrm{Pic}(X) \to \mathrm{Cl}(X)$.

It is straightforward to check that c_1 is a group homomorphism, that is, that

$$c_1(L \otimes M) = c_1(L) + c_1(M).$$

We proceed to define a map back. Given a divisor D, we define a coherent sheaf $O(D)$ on X to have sections on U given by the additive group of rational functions f such that $(f) + D \geq 0$ on U. For example, if D is an irreducible divisor, then $O(D)$ is the sheaf of rational functions which are regular outside D and which have at most a pole of order 1 along D. I claim that $O(D)$ is in fact a line bundle, for every divisor. This crucially uses the assumption that X is factorial: then every irreducible divisor D is defined locally by one function g, and so the sheaf $O(D)$ locally consists of g^{-1} times the sheaf O_X of regular functions; thus it is locally isomorphic to the sheaf O_X, as we want. More generally, for any divisor $D = \sum a_i D_i$ with D_i irreducible, the sheaf $O(D)$ is locally $g_1^{-a_1} \cdots g_r^{-a_r}$ times O_X, so it is a line bundle.

So mapping D to $O(D)$ gives a function $\mathrm{Div}(X) \to \mathrm{Pic}(X)$, which is easily seen to be a group homomorphism. Also, for a rational function f, the line bundle associated to the divisor (f) is trivial (indeed, multiplication by f gives an isomorphism from $O((f))$ to O_X), so we have a homomorphism $\mathrm{Cl}(X) \to \mathrm{Pic}(X)$. It is easy to compute that $c_1(O(D)) = D$, because the line bundle $O(D)$ comes with a rational section (the function 1) whose divisor of zeros is exactly D. So $c_1 : \mathrm{Pic}(X) \to \mathrm{Cl}(X)$ is surjective.

To show that c_1 is injective, we can argue as follows. Suppose a line bundle L has $c_1(L) = 0$ in $\mathrm{Cl}(X)$. That means that, for a rational section s of L, $(s) = (f)$ for some rational function f. So the rational section $t := s/f$ of L has $(t) = 0$. That is, t has no zeros or poles on any divisor in X. I claim that in fact t has no zeros or poles anywhere on X, so that t is a trivialization of L, as we want. To prove the claim, we can work locally on X, so we can assume L is trivial; we want to show that a rational function with no zeros or poles on divisors is in fact a unit. However, it is a general fact of commutative algebra that the zeros and poles of rational functions on normal schemes occur only on codimension-1 subvarieties; this applies to X since X is factorial. (See Hartshorne, Prop. II.6.2 and II.6.3A.) QED

4 Divisors and line bundles: examples

I wil discuss some examples of computations of the Picard group very briefly. Some of these calculations will be done later in the course, but it would be a good idea to read about these calculations now if they're not familiar.

Example 1. The divisor class group of affine space A^n over a field is trivial. This is equivalent to the theorem (by Gauss?) that the corresponding ring $k[x_1, \ldots, x_n]$ is a unique factorization domain (Hartshorne, Prop. II.6.2 for the equivalence; Lang, Algebra, Cor. V.6.3 (or books on commutative algebra) for the theorem that polynomial rings are UFDs; or Hartshorne, Prop. II.6.6 for the geometric proof.)

Example 2. The Picard group (or divisor class group) of projective space $\mathbf{P}^r$ over a field is isomorphic to $\mathbf{Z}$ for $r \geq 1$, generated by the hyperplane line bundle $O(1)$. (This is the line bundle $O(H)$ for a hyperplane H in $\mathbf{P}^n$, so that $c_1(O(1)) = H$ in the divisor class group.) We write $O(n)$ for $O(1)^{\otimes n}$ for any integer n, so any line bundle on projective space has the form $O(n)$ for a uniquely determined integer n. You should know the space of sections $H^0(\mathbf{P}^r, O(n))$ for any n, and it is useful to know what the higher cohomology looks like, too: see Hartshorne, section III.5, or Griffiths-Harris.

Example 3. Let X be a smooth projective curve over a field k. An irreducible divisor on X is a 0-dimensional subvariety, so we have defined its degree over k, which is a positive integer. Extending this by linearity gives the notion of the degree of a divisor, which is an integer. (If k is algebraically closed, a divisor on X is just a finite sum of points in $X(k)$ with integer coefficients, and the degree of the divisor is just the sum of the coefficients.) A basic theorem on algebraic curves (Hartshorne or Griffiths-Harris) says that the degree of the divisor of a rational function on a smooth projective curve is zero (that is, the number of zeros, counted with multiplicity, equals the number of poles). So the degree of a divisor is well-defined on linear equivalence classes, and so we get the notion of the degree of a line bundle on a curve,

$$\deg : \mathrm{Pic}(X) \to \mathbf{Z},$$

defined by $\deg(L) = \deg(c_1(L))$. Define $\mathrm{Pic}^0(X)$ to be the kernel of this homomorphism, the group of line bundles of degree zero.

Then $\mathrm{Pic}^0(X)$ is (the group of k-points of) the *Jacobian* of the curve X, assuming that X has a k-rational point. The Jacobian A of a curve of genus g is an abelian variety A of dimension g over k, and I am saying that $\mathrm{Pic}^0(X) = A(k)$. An abelian variety is a smooth projective variety which is also an algebraic group; it is automatically a commutative algebraic group. For example, abelian varieties of dimension 1 are just elliptic curves. An abelian variety A of dimension g over the complex numbers is a compact complex torus, meaning that it is isomorphic as a complex manifold to $\mathbf{C}^g/(\mathbf{Z}^{2g})$ for some embedding of $\mathbf{Z}^{2g}$ as a discrete subgroup of $\mathbf{C}^g$. In particular, as a topological group, it is isomorphic to $(S^1)^{2g}$.

The divisor class group is a special case of the Chow groups we will study, and the case of curves shows many phenomena which will extend to Chow groups. First, Picard groups can be big: the Picard group of a curve of genus at least 1 over the complex numbers is an uncountable abelian group, unlike the integral cohomology of the curve. Next, Picard groups contain a lot of information: the Jacobian of a curve, as an algebraic group, determines the curve up to isomorphism (the Torelli theorem), and, more generally, the detailed study of algebraic curves strongly depends on the Jacobian (see Griffiths and Harris, chapter 2). Finally, Picard groups can be hard to compute. The group Pic^0 of an elliptic curve E over $\mathbf{Q}$ is isomorphic to the group of rational points on E, and finding an algorithm for computing this is one of the main open problems of number theory, related to the problem of finiteness of the Tate-Shafarevich group and the Birch-Swinnerton-Dyer conjecture.

5 Divisors and line bundles on singular schemes

Now we consider divisors and line bundles on singular schemes.

For any variety X over k, the group of Weil divisors on X is the free abelian group on the set of codimension-one subvarieties of X. It takes more work to define the divisor of zeros of a rational function on a singular variety X. For example, let X be a curve with a node p over the complex numbers. Then a rational function on X is equivalent to a rational function on the normalization of X. The inverse image of a small neighborhood of p (in the classical topology) is a disjoint union of two smooth curves, and we define the order of f at the singular point to be the sum of the orders of f at the corresponding two points.

More generally, to define the order of a rational function f on a codimension-one subvariety D of a variety X, we again look at the local ring R of X at D. This is a one-dimensional local domain with quotient field $k(X)$. For a nonzero element f of R, we define

$$\mathrm{ord}_D(f) = l_R(R/(f)),$$

the length of $R/(f)$ as an R-module. For a local ring R, the only simple R-module is the residue field $R/m = k(D)$, and $R/(f)$ is a module of finite length, meaning that it is an extension of finitely many simple modules. The length means the total number of simple modules that appear in such a filtration, which is independent of the choice of filtration (the Jordan-Holder theorem).

For any two nonzero elements f and g of R, we have a short exact sequence

$$0 \to R/(f) \to R/(fg) \to R/(g) \to 0,$$

where the first map is multiplication by g; you can check that this sequence is exact, just using that R is a domain. Therefore $\mathrm{ord}_D(fg) = \mathrm{ord}_D(f) + \mathrm{ord}_D(g)$. It follows that we have a well-defined homomorphism from the quotient field $k(X)^*$ of R to $\mathbf{Z}$ by

$$\mathrm{ord}_D(f/g) = \mathrm{ord}_D(f) - \mathrm{ord}_D(g).$$

So we can define the divisor of a nonzero rational function f, written (f) or $\mathrm{div}(f)$, on any variety X as $\sum_D \mathrm{ord}_D(f) D$. We say that two divisors are linearly equivalent if their difference is the divisor of a rational function, and we define the divisor class group $\mathrm{Cl}(X)$ as the quotient of the group of Weil divisors by the subgroup of divisors of rational functions. We always have a homomorphism

$$c_1 : \mathrm{Pic}(X) \to \mathrm{Cl}(X)$$

defined by the divisor of any rational section of a line bundle. But this is not an isomorphism for general singular schemes X. For X normal, c_1 is injective, but it can happen that not every divisor is even locally the divisor of a rational function.

Example (Hartshorne, Example II.6.5.2). Let X be the affine quadric cone $xy = z^2$ in A^3, which is normal. Then $\mathrm{Pic}(X)$ is zero, but $\mathrm{Cl}(X) = \mathbf{Z}/2$. The divisor class group is generated by a line through the origin, say $\{(x, 0, 0)\} \subset A^3$. The class of this line D is nonzero because one can check that D is not defined by one equation in a neighborhood of the origin. But the function y on X vanishes to order 2 along D and nowhere else, so $2D = 0$ in $\mathrm{Cl}(X)$.

6 Chow groups and Borel-Moore homology

Let X be a scheme of finite type over a field k. The group $Z_r(X)$ of *algebraic cycles* of dimension r on X is the free abelian group on the set of r-dimensional closed subvarieties of X. Thus an r-cycle on X is a finite sum $\sum a_i Z_i$ where a_i are integers and each Z_i is a closed subvariety of dimension r.

For any $(r+1)$-dimensional closed subvariety W of X, and any nonzero rational function f on W, the divisor (f) is a divisor on W, hence an r-cycle on X. Say that an r-cycle α on X is *rationally equivalent* to zero if there are finitely many subvarieties W_j and rational functions f_j on W_j such that

$$\alpha = \sum \mathrm{div}(f_j) \in Z_r(X).$$

The cycles rationally equivalent to zero clearly form a subgroup, and so we can define the quotient group, the *Chow group* of cycles modulo rational equivalence:

$$CH_r(X) = Z_r(X)/(\text{cycles rationally equivalent to zero}).$$

We can write $[Z]$ to mean the class of a subvariety Z in the Chow groups.

Example: For X a scheme of dimension n, there are no subvarieties of dimension $n + 1$, and so the Chow group $CH_n(X)$ is simply the free abelian group on the set of n-dimensional irreducible components of X.

For an n-dimensional variety X, the Chow group $CH_{n-1}X$ is simply the divisor class group $\mathrm{Cl}(X)$.

For any scheme X, the subvarieties of X are the same as the subvarieties of the reduced scheme X_{red}. Therefore

$$CH_i(X_{\mathrm{red}}) \cong CH_i(X)$$

for all i. That is, the Chow groups of a scheme only depend on the underlying reduced scheme.

Over the complex numbers, we can map Chow groups to homology groups. For noncompact varieties, the relevant version of homology is Borel-Moore homology (what Bott-Tu's book Differential Forms in Algebraic Topology calls homology with closed support), and so we discuss this first. See Fulton's Intersection Theory, section 19.1, for more details on Borel-Moore homology than I can give here.

For any locally compact space X and commutative ring R, we can define Borel-Moore homology $H_i^{BM}(X, R)$ as the homology of the chain complex of locally finite singular chains. That is, instead of the chain group being the set of finite R-linear combinations of singular simplices, we allow infinite linear combinations such that every point has a neighborhood which meets only finitely many of the singular simplices. It follows that for compact spaces, Borel-Moore homology is the same thing as ordinary homology, $H_i(X, R) = H_i^{BM}(X, R)$.

We can also define Borel-Moore homology of any nice locally compact space directly in terms of ordinary homology: with any coefficients, $H_i^{BM}(X)$ is the inverse limit of the groups $H_i(X, X - K)$ where K runs over all compact subsets of X. ("Nice" locally compact spaces, for this purpose, include the spaces $X - Y$ for any finite CW-complex X and any closed subcomplex Y. That includes all the topological spaces associated to complex algebraic varieties, since every algebraic variety is a Zariski open subset of some compact variety, and every compact complex algebraic variety has a triangulation into finitely many simplices.)

In other words, Borel-Moore homology is the homology of a space "relative to infinity". For example, it follows that $H_i^{BM}(\mathbf{R}^n, \mathbf{Z})$ is $\mathbf{Z}$ if $i = n$ and zero otherwise (the homology of an n-disc relative to its boundary). More generally, if X is the interior of a manifold with boundary M, then $H_i^{BM}(X)$ is isomorphic to the relative homology $H_i(M, \partial M)$. Even more generally, for any finite CW-complex Y and closed subcomplex Z, the Borel-Moore homology of $Y - Z$ is simply the relative homology of the pair (Y, Z). Equivalently, the Borel-Moore homology of any "nice" noncompact space X (for example $Y - Z$ as above) is simply the homology of the one-point compactification of X relative to the added point. These interpretations of Borel-Moore homology apply in particular to all complex algebraic varieties.

With field coefficients, just as the dual of homology groups are cohomology groups, Borel-Moore homology groups are dual to cohomology with compact support:

$$H_c^i(X, F)^* = H_i^{BM}(X, F).$$

One major convenience of Borel-Moore homology is that it allows a good statement of Poincaré duality for noncompact manifolds X. Namely, an oriented n-manifold X (compact or not) has a fundamental class $[X]$ in $H_n^{BM}(X, \mathbf{Z})$. Also, Borel-Moore homology is a module over the cohomology ring for any locally compact space (via "cap product"), just as with usual homology:

$$\cap : H^i(X) \otimes H_j^{BM}(X) \to H_{j-i}^{BM}(X).$$

Then cap product with the fundamental class gives a Poincaré duality isomorphism for any oriented manifold X, compact or not:

$$\cap[X] : H^i(X, \mathbf{Z}) \cong H_{n-i}^{BM}(X, \mathbf{Z}).$$

In particular, every smooth complex algebraic variety of dimension n has a canonical orientation (see Griffiths-Harris, chapter 0), and so we can identify its Borel-Moore homology with cohomology.

While usual homology has a pushforward map for all continuous maps, Borel-Moore homology only has a pushforward for proper maps $f : X \to Y$, $f_* : H_i^{BM}(X) \to H_i^{BM}(Y)$, $i \geq 0$. (A continuous map of locally compact spaces is called proper if the inverse image of every compact set is compact.) Think about the definition of Borel-Moore homology to see why. This makes Borel-Moore homology with R coefficients a functor from the category of locally compact spaces and proper maps to the category of R-modules. Also, for the inclusion $f : U \to X$ of an open subset, there is a pullback map on Borel-Moore homology groups, $f^* : H_i^{BM}(X) \to H_i^{BM}(U)$. (To see why, use the interpretation of Borel-Moore homology in terms of one-point compactifications. For an open inclusion $f : U \to X$, there is no map of one-point compactifications $U^* \to X^*$, but there is an obvious map $X^* \to U^*$: think about it.)

Using these maps, we can write out the basic exact sequence of Borel-Moore homology groups, the localization sequence. For a closed subset Y of a locally compact space X, we have a long exact sequence

$$\to H_i^{BM} Y \to H_i^{BM} X \to H_i^{BM}(X - Y) \to H_{i-1}^{BM} Y \to \cdots .$$

(A closed inclusion is proper, and the first map is the associated pushforward map. The next map is the pullback map associated to an open inclusion.) For example, if X and Y are compact, then $H_i^{BM}(X - Y) = H_i(X, Y)$ and this is just the long exact sequence of relative homology. There is also a Mayer-Vietoris sequence for Borel-Moore homology,

$$\to H_i^{BM} X \to H_i^{BM} U \oplus H_i^{BM} V \to H_i^{BM} U \cap V \to H_{i-1}^{BM} X \to \cdots ,$$

for open subsets U and V whose union is X.

Using the localization sequence, we see that $H_n^{BM}(X)$ does not change if you remove a closed subcomplex of real codimension at least 2 from a CW complex X. Now every complex algebraic variety X (not necessarily smooth or compact) has singular set of complex codimension at least 1, hence of real codimension 2. Therefore, if X has complex dimension n, we have $H_{2n}^{BM}(X, \mathbf{Z}) \cong \mathbf{Z}$, with a canonical generator called the *fundamental class* $[X]$ of X. This gives a natural way to write down elements of the Borel-Moore homology of any complex variety: just take the fundamental class of any closed subvariety. (Note that a closed subvariety of a noncompact variety will usually be nonompact, whence the need to use Borel-Moore homology rather than the usual homology groups.)

For any complex variety X, taking the fundamental classes of subvarieties gives a homomorphism

$$Z_i(X) \to H_{2i}^{BM}(X, \mathbf{Z})$$

for $i \geq 0$. Fulton checks (section 19.1) that two rationally equivalent cycles are homologous; we may check this later. It follows that we have a homomorphism, called the *cycle map*, from the Chow groups of any complex variety to Borel-Moore homology:

$$CH_i(X) \to H_{2i}^{BM}(X, \mathbf{Z}).$$

7 Pushforward of cycles

Let $f : X \to Y$ be a proper morphism of schemes over k. (See Hartshorne for the definition and properties of a proper morphism of schemes. Conveniently, a morphism of schemes $X \to Y$ over the complex numbers is proper if and only if the corresponding map of topological spaces $X(\mathbf{C}) \to Y(\mathbf{C})$ (with the classical topology) is proper.) For any closed subvariety V of X, the image $W = f(V)$ is a closed subvariety of Y by properness of f. The surjective morphism $V \to W$ gives an inclusion of rational function fields $k(W) \subset k(V)$. Here W has dimension at most that of V, and if it has the same dimension, then the field $k(V)$ is a finite extension of $k(W)$. Define

$$\deg(V/W) = \begin{cases} [k(V) : k(W)] & \text{if } \dim(W) = \dim(V) \\ 0 & \text{if } \dim(W) < \dim(V). \end{cases}$$

Here $[k(V) : k(W)]$ denotes the degree of the field extension, that is, the dimension of $k(V)$ as a vector space over $k(W)$.

For k perfect, we can give a more geometric interpretation of the degree $V \to W$: extend k to its algebraic closure, and then the degree is simply the number of points in the inverse image of most points of W. (There is a dense open subset of W of points whose inverse image has order equal to the degree of $V \to W$ in the above algebraic sense.) Here I am thinking of the case where V and W have the same dimension; if W has smaller dimension, then the inverse image of most points in W has dimension greater than zero. Also, over the complex numbers, the degree of $V \to W$ is given by the pushforward on homology groups: if $n = \dim(V)$, then $f_* : H_{2n}^{BM}(V, \mathbf{Z}) = \mathbf{Z} \to H_{2n}^{BM}(W, \mathbf{Z})$ maps the fundamental class $[V]$ to $\deg(V/W)$ times the fundamental class $[W]$.

For a proper morphism $f : X \to Y$ of schemes over k and a subvariety V of X, we define

$$f_*(V) = \deg(V/W)W$$

in $Z_*(Y)$. Extending this by linearity, we get a *pushforward* homomorphism

$$f_* : Z_r(X) \to Z_r(Y)$$

These pushforwards are functorial; that is, $(gf)_* = g_* f_*$ for proper maps f and g, by the multiplicativity of degrees of field extensions.

Theorem 7.1 *If $f : X \to Y$ is a proper morphism, and α is an algebraic cycle on X which is rationally equivalent to zero, then $f_*(\alpha)$ is rationally equivalent to zero.*

As a result, a proper morphism determines a homomorphism on Chow groups,

$$f_* : CH_i X \to CH_i Y,$$

making the Chow groups functorial for proper morphisms of schemes over k. My proof is a variant, I hope slightly easier to understand, of Fulton's (Theorem 1.4 and Appendix A).

Proof. We can assume that α is an i-cycle for some $i \geq 0$. We can assume that α is the divisor of a rational function on a single $(i+1)$-dimensional subvariety of X. We can replace X by this subvariety, and we can replace Y by the image of that subvariety; thus Y is a variety and $f : X \to Y$ is surjective. The theorem then follows from the following more explicit lemma.

Lemma 7.2 *Let $f : X \to Y$ be a proper surjective morphism of varieties over k, and let $r \in k(X)^*$. Then*

(a) $f_*(div(r)) = 0$ if $\dim(Y) < \dim(X)$.

(b) $f_*(div(r)) = div(N(r))$ if $\dim(Y) = \dim(X)$.

In (b), $k(X)$ is a finite field extension of $k(Y)$, and the *norm* $N : k(X)^* \to k(Y)^*$ (as usual in field theory) is defined by taking $N(r)$ to be the determinant of multiplication by r on $k(X)$, viewed as a $k(Y)$-vector space. Note that even part (a) is not completely obvious: applied to the map from a projective curve to a point, (a) is the theorem that the number of zeros equals the number of poles (counted with multiplicities) for a rational function on a projective curve.

Proof of (b). We have a surjective morphism $f : X \to Y$ between varieties of the same dimension. Let S be the closed subset of Y over which the fibers of f have dimension at least 1. Then S has codimension at least 2: if it had codimension 1, then its inverse image would have to be all of X, contradicting surjectivity of f. Now we have to show that every codimension-1 subvariety D of Y has the same coefficient in $f_*(\text{div}(r))$ as in $\text{div}(N(r))$. Neither one changes if we replace Y by $Y - S$, and X by the inverse image of that. Thus we have arranged that $f : X \to Y$ is proper and has 0-dimensional fibers, so it is finite.

Let $\tilde{X}$ and $\tilde{Y}$ be the normalizations of X and Y. We have a commutative diagram

$$\begin{array}{ccc} \tilde{X} & \xrightarrow{\pi} & X \\ \downarrow{\scriptstyle \tilde{f}} & & \downarrow{\scriptstyle f} \\ \tilde{Y} & \xrightarrow{\pi_Y} & Y. \end{array}$$

Here $\pi : \tilde{X} \to X$ is birational, so we can view r as a rational function on $\tilde{X}$. Suppose we can prove (b1) that $\pi_*\text{div}(r) = \text{div}(r)$ for the normalization $\pi : \tilde{X} \to X$ of any variety X, and (b2) that statement (b) holds for the finite morphism $\tilde{f} : \tilde{X} \to \tilde{Y}$ of normal varieties. Then (b) would be proved, since we have $f_*\text{div}(r) = f_*\pi_*\text{div}(f) = (\pi_Y)_*\tilde{f}_*\text{div}(r) = (\pi_Y)_*\text{div}(N(r)) = \text{div}(N(r))$.

To show (b1) that $\pi_*\text{div}(r) = \text{div}(r)$ for the normalization $\pi : \tilde{X} \to X$, we have to show that each irreducible divisor D in X has the same coefficient on the two sides. Let A be the local ring of X at the generic point of D, so A is a 1-dimensional local ring, and B the normalization (integral closure in the quotient field) of A, which may have several maximal ideals m_i corresponding to the irreducible divisors V_i in $\tilde{X}$ that map onto D. We have to show that

$$\sum_i \text{ord}_{V_i}(r)[k(V_i) : k(D)] = \text{ord}_D(r).$$

Since order function is a homomorphism, it suffices to prove this when r is in A. I claim that the left side equals the length of B/rB as an A-module. Indeed, B/rB is the direct sum of B_i/rB_i where B_i is the local ring of B (or of $\tilde{X}$) at the divisor V_i. We defined $\text{ord}_{V_i}(r)$ to be the length of B_i/rB_i as a B_i-module, and considering its length as an A-module multiplies this by the degree of the residue field extension, $[k(V_i) : k(D)]$, as we want. (It suffices to check this for the unique B_i-module of length 1, namely the residue field $k(V_i)$, for which it is clear.)

On the other hand, the right side of the above equation is the length of A/rA as an A-module. So (b1) will follow if we can show that A/rA and B/rB have the same length as A-modules. This is not at all obvious, because the obvious map $A/rA \to B/rB$ need not be an isomorphism.

What we do is to consider the cokernel M of $A \to B$ as an A-module. Thus we have an exact sequence of A-modules (remember that B is the integral closure of A in its quotient field $k(X)$):

$$0 \to A \to B \to M \to 0.$$

The crucial point is that M is an A-module of finite length. Indeed, the inclusion $A \to B$ becomes an isomorphism after tensoring with the quotient field, and A is a 1-dimensional ring, so the cokernel M is supported on a 0-dimensional closed subset of $\mathrm{Spec}(A)$, which means (since M is also a finitely generated A-module) that it has finite length as an A-module.

Then we tensor the above exact sequence of A-modules with A/rA, which gives a long exact sequence:

$$\cdots \to \mathrm{Tor}_1^A(B, A/rA) \to \mathrm{Tor}_1^A(M, A/rA) \to A \otimes_A A/rA \to B \otimes_A A/rA \to M \otimes_A A/rA \to 0.$$

Using the obvious free resolution

$$0 \longrightarrow A \xrightarrow[r]{} A \longrightarrow A/rA \longrightarrow 0$$

(remember that A is a domain and r is not zero), we compute that $\mathrm{Tor}^1_A(N, A/rA)$ is equal to the r-torsion subgroup $N[r] := \{x \in N : rx = 0\}$ for any A-module N. But B is an integral domain, so $B[r] = 0$, and our exact sequence becomes:

$$0 \to M[r] \to A/rA \to B/rB \to M/rM \to 0.$$

These are all finite-length A-modules (although A and B are themselves not of finite length), and so the alternating sum of their lengths is zero. So to prove that A/rA and B/rB have the same length, it suffices to show that $M[r]$ and M/rM have the same length. For that, we use the obvious exact sequence

$$0 \longrightarrow M[r] \longrightarrow M \xrightarrow[r]{} M \longrightarrow M/rM \longrightarrow 0,$$

in which all four modules have finite length. So the alternating sum of their lengths is zero, which says that the A-modules $M[r]$ and M/rM have the same length, as we want. Thus A/rA and B/rB have the same length, and (b1) is proved.

It remains to prove (b2), that is, that statement (b) holds for a finite surjective morphism $f : X \to Y$ between normal varieties of the same dimension. Let A be the local ring of Y at a given irreducible divisor D. Since f is finite, it is affine, and so the inverse image of $\mathrm{Spec}(A)$ in X has the form $\mathrm{Spec}(B)$ for some finite A-algebra B. Here B is the semi-local ring of X at the codimension-1 subvarieties V_i that map onto D. We need to show that

$$\sum \mathrm{ord}_{V_i}(r)[k(V_i) : k(D)] = \mathrm{ord}_D(N(r)).$$

Since the norm and the order function are homomorphisms, it suffices to prove this when r is in B. By the same argument as in (b1), the left side equals the length of B/rB as an A-module.

It remains to show that $l_A(B/rB) = \mathrm{ord}_D(N(r))$. Since Y is normal, the 1-dimensional local ring A of Y at the generic point of the divisor D is normal and thus a discrete valuation ring. Since B is a domain, it is torsion-free as an A-module. Also, $X \to Y$ is finite and so B is a finitely generated A-module; by the structure of finitely generated modules over a dvr (as more generally over a pid), B is a finitely generated free A-module. So the statement we want follows from the general statement: for any finitely generated free module M over a discrete valuation ring A and any injective endomorphism r of M, the length of M/rM as an A-module is equal to $\mathrm{ord}_A(\det(r))$. But this is clear from the theory of principal divisors for modules over a principal ideal domain, which says that one can choose bases $e_1, \ldots, e_d$ and $f_1, \ldots, f_d$ for M in which the matrix for r, going from M with basis e_i to M with basis f_i is diagonal, say $(c_1, \ldots, c_d)$. Then

$$l_A(M/rM) = l_A(\oplus A/(c_i)) = \sum \mathrm{ord}_A(c_i) = \mathrm{ord}_A(\prod c_i) = \mathrm{ord}_A(\det r),$$

where the last equality uses that a matrix in $GL(n, A)$ (corresponding to the change between the two bases e_i and f_i for M) has determinant in A^*, hence with ord_A equal to 0. Thus (b) is proved.

Proof of (a). If $\dim(Y)$ has dimension less than $\dim(X) - 1$, then this is trivial ($f_*(\mathrm{div}(r))$ is a cycle of dimension greater than $\dim(Y)$, so it is zero). So we can assume that $\dim(Y) = \dim(X) - 1$. Here $f_*(\mathrm{div}(r))$ is a multiple of $[Y]$ with coefficient

$$\sum \mathrm{ord}_V(r)[k(V) : k(Y)],$$

the sum being over all codimension-one subvarieties V of X which map onto Y. We can replace Y by its function field $K := k(Y)$ and X by the generic fiber of $X \to Y$, that is, the fiber product $X \times_Y \mathrm{Spec}(k(Y))$. Thus X is a proper curve over the field K, and statement (a) is the statement that the number of zeros minus the number of poles of a nonzero rational function r on X is zero.

That is a standard fact, but anyway here is one proof. First, we reduce to the case where the curve X is regular, as follows. Let $\pi : \tilde{X} \to X$ be the normalization, so we can also think of r as a rational function on $\tilde{X}$. The singular locus of a normal scheme has codimension at least 2, so the curve $\tilde{X}$ over K is regular. By part (b) above, applied to the proper morphism π, we know that $\pi_*(\mathrm{div}(r)) = \mathrm{div}(r)$. Since the proper pushforward is functorial at the level of cycles, we have $f_*\mathrm{div}(r) = f_*\pi_*\mathrm{div}(r) = \tilde{f}_*\mathrm{div}(r)$, where $\tilde{f}$ is the composition $\tilde{X} \to X \to \mathrm{Spec}(K)$. Thus we have reduced to showing that $f_*\mathrm{div}(r) = 0$ when $f : X \to \mathrm{Spec}(K)$ is a regular proper curve. In other words, we have to show that the number of zeros equals the number of poles, correctly counted, on a regular proper curve.

We can think of r as a rational map from the regular proper curve X to $\mathbf{P}^1$. Using that the local rings of X are discrete valuation rings, it is easy to check that this is in fact a morphism $X \to \mathbf{P}^1_K$. (More generally, a rational map from a regular scheme to a proper scheme over K is defined outside a subset of codimension at least 2.) If r is constant, then $\mathrm{div}(r) = 0$ and we are done, so we can assume r is not constant; then it is clearly a finite morphism $r : X \to \mathbf{P}^1_K$ of some degree d. It suffices to show that the number of zeros and poles of r (counted as above) are both equal to d. Look at the number of zeros, the argument for poles being the same. Let A be the local ring of $\mathbf{P}^1_K$ at the point 0 and $\mathrm{Spec}(B)$ the inverse image of $\mathrm{Spec}(A)$ in X (so B is the semi-local ring of X at the points mapping to 0). Since A is a dvr and B is a finitely generated torsion-free A-module, B is a free module over A, and it has rank d since $\mathrm{Frac}(B) = K(X)$ has dimension d over $\mathrm{Frac}(A) = K(\mathbf{P}^1)$. Therefore (writing t for the standard coordinate function in A) $B/rB = B \otimes_A A/tA$ has dimension d over $A/tA = K$, where we think of our function r as an element of the ring B. But the dimension of B/rB as a K-vector space is the sum over the 0-dimensional subvarieties p in X mapping to 0 of $\mathrm{ord}_p(r)$ times the dimension of the field $H^0(p, O)$ as a K-vector space, as we want. QED

Example: For a proper scheme X over k, the pushforward homomorphism associated to $f : X \to \mathrm{Spec}(k)$,

$$f_* : CH_0(X) \to CH_0(\mathrm{Spec}(k)) = \mathbf{Z},$$

gives the degree of a 0-cycle on X, as defined earlier. This homomorphism is also written as $\int_X$. (Think of the Poincaré duality isomorphism $H_0(X, \mathbf{Z}) \cong H^{2n}(X, \mathbf{Z})$ for a smooth compact complex n-fold X, and view $H^{2n}(X, \mathbf{R})$ in terms of de Rham cohomology; then taking the degree of a 0-cycle amounts to taking the integral of a top-dimensional C^∞ differential form.)

More generally, we define $\int_X$ to be zero on cycles on positive dimension, so it becomes a group homomorphism

$$\int_X : CH_*(X) \to \mathbf{Z}$$

for a proper scheme X over a field k.

Make sure you understand why the pushforward on Chow groups is not well-defined for non-proper morphisms. For example, take the map $f_* : A^1_k \to \mathrm{Spec}(k)$. The origin on A^1 is rationally equivalent to 0, as you can check: intuitively, we can move the origin off to infinity. But the pushforward of the origin to the point $\mathrm{Spec}(k)$ is a point, which is nonzero in $CH_0(\mathrm{point}) = \mathbf{Z}$.

8 Cycles of subschemes

Let X be any scheme of dimension at most n over k. Let X_i be the n-dimensional irreducible components of X. The local rings O_{X,X_i} of X at the subvarieties X_i are zero-dimensional (also called artinian rings). So O_{X,X_i} has finite length as a module over itself, and we define the *geometric multiplicity* m_i of X_i in X to be this length:

$$m_i = l_{O_{X,X_i}}(O_{X,X_i}).$$

The *(fundamental) cycle* of X is the n-cycle

$$[X] = \sum_i m_i [X_i].$$

This is a cycle in $Z_n(X)$, and we also write $[X]$ for its image in $CH_n(X)$. (Since X has dimension at most n, these two groups are the same, both being the free abelian group on the classes $[X_i]$.)

More generally, if X is a closed subscheme of dimension at most n of a scheme Y, then $Z_*(X) \subset Z_*(Y)$, and so we can view $[X]$ as an n-cycle on Y. Again, we also write $[X]$ for the class of this cycle in $CH_n(Y)$. A slight subtlety is that this definition depends on the number n: a subscheme of dimension at most n is also a subscheme of dimension at most $n+1$, but as such its class in $CH_{n+1}Y$ is zero. Usually it should be clear what dimension we have in mind.

Example: The cycle class of the subscheme $x^2(x+y)=0$ in A^2 is 2 times the line $x=0$ plus the line $x+y=0$. The cycle class of the subscheme $\{x^3=0, xy=0, y^3=0\}$ in A^2 is 5 times the point $(0,0)$, because the ring $k[x,y]/(x^3,xy,y^3)$ has a basis as a k-vector space given by the monomials $1, x, x^2, y, y^2$ and hence has length 5 as a module over itself. (As it happens, these cycles represent zero in the Chow groups of A^2, but similar cycles on a more interesting variety (even $\mathbf{P}^2$) could be nonzero in Chow groups.)

Example: Let X be a variety of dimension n, and let $f : X \to \mathbf{P}^1$ be a dominant morphism. Let $0 = [1,0]$ and $\infty = [0,1]$ be the usual points on $\mathbf{P}^1$. The inverse image of a closed subscheme is a closed subscheme in a natural way (given by pulling back the equations of the given subscheme), so we have $(n-1)$-dimensional subschemes $f^{-1}(0)$ and $f^{-1}(\infty)$ of X. The point is that our definition of the cycle class of these subschemes makes

$$[f^{-1}(0)] - [f^{-1}(\infty)]$$

equal to the $(n-1)$-cycle $(f) = \mathrm{div}(f)$ on X. That is, the multiplicity of a divisor in the cycle $[f^{-1}(0)]$, say, is just the integer $\mathrm{ord}_D(f)$ (when this is positive).