Problem sheet 1, PCMI

Burt Totaro

July 2024, Park City, Utah

(1) Consider the real affine curve $S^1_{\mathbf{R}} = \{x^2 + y^2 = 1\}$ in $A^2_{\mathbf{R}}$. Compute the Chow groups of $S^1_{\mathbf{R}}$. Also, compute the Chow groups of $S^1_{\mathbf{C}}$ and the homomorphism $CH_0(S^1_{\mathbf{R}}) \to CH_0(S^1_{\mathbf{C}})$.

(2) Compute the Chow groups of $\mathbf{P}^2_{\mathbf{C}}$ minus a smooth conic. (A *conic* over a field k is a curve of degree 2 in $\mathbf{P}^2_k$. So it is given by $f = 0$ for some homogenous polynomial $f(x, y, z)$ of degree 2 over k. By my conventions, a curve is irreducible; so we are assuming that the polynomial f is irreducible.)

(3) Show that a conic X over a field k with a k-rational point p is isomorphic to $\mathbf{P}^1_k$. (Hint: consider the family of lines through p, and how they meet X.)

(4) Let X be a smooth conic over a field k, with algebraic closure $\overline{k}$. Show that the degree homomorphism $CH_0(X_{\overline{k}}) \to \mathbf{Z}$ is an isomorphism. For any field k, show that the degree homomorphism $CH_0(X_k) \to \mathbf{Z}$ is injective, with image $\mathbf{Z}$ if X has a k-rational point and $2\mathbf{Z}$ otherwise.

(5) Let X be a smooth scheme over a field k. For a vector bundle E of rank r and a line bundle L on X, show that the Chern classes of the tensor product $E \otimes L$ in the Chow ring of X are given by:

$$c(E \otimes L) = \sum_{i=0}^{r} c_i(E)(1 + c_1 L)^{r-i}.$$

(6) Compute the Chern classes of $\mathbf{P}^n_k$ (that is, of the tangent bundle) in the Chow ring. Read off the Chern numbers c_1^2 and c_2 for $\mathbf{P}^2_k$.

(7) The Chow group of 0-cycles is known to be birationally invariant among smooth proper varieties over a field. Show that it is not birationally invariant among projective varieties over $\mathbf{C}$ which may be singular. Ideally, show this failure among normal projective varieties.