PROBLEM SET

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Abstract. This is the problem set for the authors' 2024 PCMI minicourse on Massey products in Galois cohomology.

Contents

1. Quadratic forms

Reference: [3].

1. Let V be a vector space over a field F of characteristic not 2. A *quadratic form on V* is a map $q : V \to F$ such that $q(av) = a^2 q(v)$ for all $a \in F$ and $v \in V$ and the associated map

$$b_q : V \times V \to F \quad (v, w) \mapsto q(v + w) - q(v) - q(w)$$

is a bilinear form. A quadratic form q is called *nondegenerate* (or *nonsingular*) if the bilinear form b_q is nondegenerate; cf [3, Proposition 1.2].

Two quadratic forms are said to be *isomorphic* (or *isometric*) if there exists an isomorphism of the underlying vector spaces which respects the bilinear forms; see [3, p. 4]. The *orthogonal sum* $g \perp h$ of two quadratic forms g and h is defined in a natural way; see [3, p. 6].

For all $a_1, \ldots, a_n \in F^\times$, we write $\langle a_1, a_2, \ldots, a_n \rangle$ for the quadratic form

$$a_1 x_1^2 + a_2 x_2^2 + \ldots + a_n x_n^2.$$

Prove that every nondegenerate quadratic form is isomorphic to $\langle a_1, a_2, \ldots, a_n \rangle$ for some $a_i \in F^\times$.

2. The form $\mathbb{H} = \langle 1, -1 \rangle$ is the *hyperbolic plane*. A form isomorphic to the orthogonal sum $\mathbb{H}^{\perp n}$ of $n > 0$ copies of the hyperbolic plane is called *hyperbolic*. A nondegenerate quadratic form $q : V \to F$ is *isotropic* if $q(v) = 0$ for some nonzero $v \in V$. Prove that a nondegenerate quadratic form q is isotropic if and only if $q \simeq \mathbb{H} \perp q'$ for some quadratic form q'.

3. Two nondegenerate quadratic forms g and h are called *Witt equivalent* if $g \perp \mathbb{H}^{\perp n} \simeq h \perp \mathbb{H}^{\perp m}$ for some $n, m \geq 0$. The set of equivalence classes $[q] \in W(F)$ of nondegenerate quadratic forms q over F is endowed with the two operations

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$[g]+[h]=[g\perp h]$ and $[g]\cdot[h]=[g\otimes h]$; see [3, I.§6, p. 17] for the definition of the Kronecker product $g\otimes h$. Prove that $W(F)$ is a commutative ring, called the *Witt ring* of F.

For simplicity, we will write q for $[q]$ in $W(F)$.

4. Show that the Cancellation Law $f\perp g\simeq f\perp h\Rightarrow g\simeq h$ (see [3, I.§4, Theorem 4.2]) implies that two nondegenerate quadratic forms g and h are isomorphic if and only if $g=h$ in $W(F)$ and $\dim(g)=\dim(h)$.

5. Let $a,b\in F^\times$. The form

$$\langle\langle a,b\rangle\rangle=\langle 1,-a,-b,ab\rangle=\langle 1,-a\rangle\otimes\langle 1,-b\rangle$$

is called a 2-*fold Pfister form.* Let $Q=(a,b)$ be the quaternion algebra, i.e., Q is a 4-dimensional algebra with basis $\{1,i,j,k\}$ and multiplication table $k=ij=-ji$, $i^2=a$ and $j^2=b$. Prove that if $Q=(a,b)$ is a quaternion algebra then the reduced norm quadratic form Nrd_Q on Q is isomorphic to $\langle\langle a,b\rangle\rangle$. Show that if Q is split, then Nrd_Q is hyperbolic, otherwise, Nrd_Q is anisotropic. (See [3, III.§1 and §2] or [1, Chapter 1] for the definitions.)

6. Prove that the set $D(q)$ of nonzero values of $q=\langle\langle a,b\rangle\rangle$ is closed under multiplication. Prove that $dq\simeq q$ for every $d\in D(q)$.

7. If $q=\langle\langle a,b\rangle\rangle$, we write q° for the form $\langle a,b,-ab\rangle$, thus $q=\langle 1\rangle\perp(-q^\circ)$. Let Q be a quaternion algebra and let $Q^\circ\subset Q$ be the subspace of pure quaternions. Consider the 3-dimensional quadratic form h on Q° given by $h(x)=x^2$. Prove that $h\simeq(\mathrm{Nrd}_Q)^\circ$.

8. Two form g and h are called *similar* (or *similar in* $W(F)$) if $h\simeq ag$ (respectively, $h=ag$ in $W(F)$) for some $a\in F^\times$. Let q_1 and q_2 be two 2-fold Pfister forms. Suppose that for $a_1,a_2\in F^\times$, the forms a_1q_1 and a_2q_2 have a common nonzero value. Prove that $a_1q_1-a_2q_2$ is similar to $q_1-q_2=q_2^\circ-q_1^\circ$ in $W(F)$.

Let K/F be a quadratic field extension.

9. Prove that for every $x,y\in K\setminus F$ such that $x/y\notin F$ there exist nonzero $a,b\in F$ such that $ax+by=1$. Prove that $\langle\langle ax,by\rangle\rangle=0$ and

$$\langle\langle a,b\rangle\rangle+\langle\langle x,y\rangle\rangle=\langle\langle a,y\rangle\rangle+\langle\langle b,x\rangle\rangle$$

in $W(K)$.

10. Let $s:K\to F$ be a nonzero F-linear map such that $s(1)=0$. Prove that s is unique up to an F-multiple. Prove that if h is a (nondegenerate) quadratic form over K, then $s_*(h):=s\circ h$ is a (nondegenerate) quadratic form over F of dimension $2\dim(h)$. Show that the map $s_*:W(K)\to W(F)$ is a group homomorphism (called the *transfer* map). Prove that $s_*(\mathrm{Im}(W(F)\to W(K))=0$.

11. Prove that $s_*(\langle\langle b,x\rangle\rangle)$ is similar to $\langle\langle b,N_{K/F}(x)\rangle\rangle$ and $s_*(\langle\langle a,y\rangle\rangle)$ is similar to $\langle\langle a,N_{K/F}(y)\rangle\rangle$ in $W(F)$.

12. Prove that

$$s_*\langle\langle x,y\rangle\rangle=s_*\langle\langle a,y\rangle\rangle+s_*\langle\langle b,x\rangle\rangle$$

in $W(F)$. Show that the forms $s_*\langle\langle a,y\rangle\rangle$ and $-s_*\langle\langle b,x\rangle\rangle$ have a common nonzero value in F.

13. Deduce from Problem 8 that the 6-dimensional forms $s_*(\langle\langle x,y\rangle\rangle^\circ)$ and

$$\langle\langle a, N_{K/F}(y)\rangle\rangle^\circ) \perp (-\langle\langle b, N_{K/F}(x)\rangle\rangle^\circ)$$

are similar. (Remark: A *biquaternion algebra* is the tensor product of two quaternion algebras, $A = B \otimes_F C$. The 6-dimensional form $(\mathrm{Nrd}_B)^\circ \perp -(\mathrm{Nrd}_C)^\circ$ is called an *Albert form* of A; see [2, §16]. It depends on the decomposition of A into tensor product of two quaternion algebras, but every two Albert forms of A are similar. One can restate the exercise as follows: Prove that for every quaternion algebra Q over K, the form $s_*(\mathrm{Nrd}_Q)^\circ$ is similar to an Albert form of the biquaternion algebra $N_{K/F}(Q)$.)

14. Let Q be a quaternion algebra over K. Prove that the algebra $N_{K/F}(Q)$ is split if and only if the 6-dimensional form $s_*(\mathrm{Nrd}_Q)^\circ$ is hyperbolic.

2. Group cohomology

References: [5], [4, Chapter 1].

1. Let Γ be a profinite group and let M be a Γ-module considered as a discrete topological space. For an integer $n \geq 0$ write $C^n(\Gamma, M)$ for the abelian group of all continuous maps (*n-cochains*) $\Gamma^n \to M$. Consider the homomorphisms

$$d^n : C^n(\Gamma, M) \to C^{n+1}(\Gamma, M)$$

defined by the formula

$$\begin{aligned} d^n(\varphi)(x_1, \dots, x_{n+1}) = x_1\varphi(x_2, \dots, x_{n+1}) + \\ \sum_{i=1}^{n} (-1)^i \varphi(x_1, \dots, x_{i-1}, x_i x_{i+1}, \dots, x_{n+1}) + \\ (-1)^{n+1}\varphi(x_1, \dots, x_n). \end{aligned}$$

Show that $d^{n+1} \circ d^n = 0$ for all n. Define the following groups:

$Z^n(\Gamma, M) = \mathrm{Ker}(d^n)$ the group of *n-cocycles* of Γ with values in M,

$B^n(\Gamma, M) = \mathrm{Im}(d^{n-1})$ the group of *n-coboundaries* of Γ with values in M,

$H^n(\Gamma, M) = Z^n(\Gamma, M)/B^n(\Gamma, M)$ the *n-th cohomology group* of Γ with values in M.

2. Show that

$$H^0(\Gamma, M) = M^\Gamma := \{m \in M \mid xm = m \text{ for all } x \in \Gamma\}.$$

3. Prove that an exact sequence $0 \to M \to N \to P \to 0$ of Γ-modules yields an infinite exact sequence

$$\dots \to H^n(\Gamma, M) \to H^n(\Gamma, N) \to H^n(\Gamma, P) \to H^{n+1}(\Gamma, M) \to \dots$$

4. Show that if Γ acts trivially on M, the group $H^1(\Gamma, M)$ is equal to the group of all continuous homomorphisms $\Gamma \to M$.

5. Prove that if M is a Γ-module and $\Gamma' \subset \Gamma$ is a (closed) subgroup, then the restriction map $C^n(\Gamma, M) \to C^n(\Gamma', M)$ yields the *restriction* homomorphism

$$\mathrm{res}_{\Gamma/\Gamma'} : H^n(\Gamma, M) \to H^n(\Gamma', M).$$

If Γ' is an open subgroup of Γ (and therefore, of finite index), there is the *corestriction* homomorphism

$$\mathrm{cor}_{\Gamma/\Gamma'} : H^n(\Gamma', M) \to H^n(\Gamma, M).$$

6. Prove that if Γ' is a normal subgroup of Γ, then the group $M^{\Gamma'}$ has the natural structure of a Γ/Γ'-module and the natural map $C^n(\Gamma/\Gamma', M^{\Gamma'}) \to C^n(\Gamma, M)$ yields the *inflation* homomorphism

$$\mathrm{inf}_{\Gamma/\Gamma'} : H^n(\Gamma/\Gamma', M^{\Gamma'}) \to H^n(\Gamma, M).$$

Prove that the inflation homomorphisms yield an isomorphism

$$\operatorname{colim} H^n(\Gamma/\Gamma', M^{\Gamma'}) \xrightarrow{\sim} H^n(\Gamma, M),$$

where the colimit (direct limit) is taken over all open normal subgroups $\Gamma' \subset \Gamma$.

7. Let $\varphi \in Z^n(\Gamma, N)$ and $\psi \in Z^m(\Gamma, M)$ be two cocycles. Prove that the function $\varphi \cup \psi : \Gamma^{n+m} \to N \otimes_{\mathbb{Z}} M$ defined by the formula

$$(\varphi \cup \psi)(x_1, \dots, x_n, y_1, \dots, y_m) = \varphi(x_1, \dots, x_n) \otimes x_1 \cdots x_n \psi(y_1, \dots, y_m)$$

is a $(n+m)$-cocycle. Prove that this construction yields a well-defined *cup-product* bilinear map

$$\cup : H^n(\Gamma, N) \times H^m(\Gamma, M) \to H^{n+m}(\Gamma, N \otimes_{\mathbb{Z}} M).$$

Prove the *projection formula*:

$$\mathrm{cor}_{\Gamma/\Gamma'}(\varphi \cup \mathrm{res}_{\Gamma/\Gamma'}(\psi)) = \mathrm{cor}_{\Gamma/\Gamma'}(\varphi) \cup \psi,$$

where Γ' is an open subgroup of Γ, $\varphi \in H^n(\Gamma', N)$ and $\psi \in H^m(\Gamma, M)$.

8. Let F be a field, let F_{sep} be a separable closure of F and $\Gamma = \Gamma_F = \mathrm{Gal}(F_{\mathrm{sep}}/F)$. The multiplicative group $F_{\mathrm{sep}}^\times$ of F_{sep} is a Γ-module. We write

$$H^n(F, F_{\mathrm{sep}}^\times) := H^n(\Gamma, F_{\mathrm{sep}}^\times).$$

Prove that $H^0(F, F_{\mathrm{sep}}^\times) = F^\times$ and $H^n(F, F_{\mathrm{sep}}^\times) \simeq \operatorname{colim} H^n(G, L^\times)$, where the colimit is taken over all finite Galois field sub-extensions L/F of F_{sep}/F and $G = \mathrm{Gal}(L/F)$.

9. Let L/F be a finite field extension with Galois group G. Let $l : G \to L$ be a map such that $\sum_{\tau \in G} l(\tau) \cdot \tau(x) = 0$ for all $x \in L$. Prove that $l(\tau) = 0$ for all $\tau \in G$.

Let $l : G \to L^\times$ be a 1-cocycle. Show that there is $x \in L$ such that

$$y := \sum_{\tau \in G} l(\tau) \cdot \tau(x) \neq 0.$$

Prove that $l(\sigma) = y/\sigma(y)$ for all $\sigma \in G$. Deduce that $H^1(G, L^\times) = 1$ and $H^1(F, F_{\mathrm{sep}}^\times) = 1$. This is *Hilbert's Theorem 90.*

10. Prove that the sequence of Γ-modules $0 \to \mathbb{Z}/2\mathbb{Z} \to F_{\mathrm{sep}}^\times \to F_{\mathrm{sep}}^\times \to 1$, where the first map takes $1 + 2\mathbb{Z}$ to -1 in $F_{\mathrm{sep}}^\times$ and the second map takes x to x^2. Deduce that there is a canonical *Kummer* isomorphism

$$\mathrm{Hom}_{cont}(\Gamma, \mathbb{Z}/2\mathbb{Z}) = H^1(F, \mathbb{Z}/2\mathbb{Z}) \simeq F^\times/F^{\times 2}.$$

11. For an element $a \in F^\times$ write $\chi_a : \Gamma \to \mathbb{Z}/2\mathbb{Z}$ for the character of Γ corresponding to $aF^{\times 2}$ under the Kummer isomorphism. If χ_a is a nontrivial character,

show that the subfield of all $\mathrm{Ker}(\chi_a)$-invariant elements in F_{sep} is isomorphic to $F_a = F(a^{1/2})$.

3. Brauer group

Reference: [1].

1. Let A and B be two F-algebras. Prove that

$$M_n(A) \otimes_F M_m(B) \simeq M_{nm}(A \otimes_F B).$$

2. Let $Z(A)$ be the center of an F-algebra A. Prove that

$$Z(A \otimes_F B) = Z(A) \otimes_F Z(B)$$

for every two F-algebras A and B. In particular, if both A and B are central F-algebras, then so is $A \otimes_F B$.

3. A finite dimensional F-algebra A is called *simple* if A has no proper two-sided ideals. By *Wedderburn's Theorem* [1, Theorem 2.1.3] and Problem 2, every central simple F-algebra A is isomorphic to $M_k(D)$, where D is a central division F-algebra. The algebra D is uniquely determined by A (up to isomorphism) as the endomorphism algebra of a (unique) simple right A-module; cf. [1, Lemma 2.1.6]. Let A and B be two simple F-algebras. Prove that if A is central, then $A \otimes_F B$ is simple. In particular, if A and B are central simple F-algebras, then so is $A \otimes_F B$.

4. Two finite-dimensional central simple F-algebras A and B are called *equivalent* if $M_n(A) \simeq M_m(B)$ for some integers $n, m > 0$. The set $\mathrm{Br}(F)$ of equivalence classes $[A]$ of central simple F-algebras form the *Brauer group* via the (additively written) operation

$$[A] + [B] = [A \otimes_F B].$$

Prove that $-[A]$ is the class of the opposite algebra A^{op} (see [1, p. 32] for the definition of A^{op}).

If A is a central simple algebra, we will often write A instead of $[A]$.

5. Prove that two central simple F-algebras A and B are isomorphic if and only if $A = B$ in $\mathrm{Br}(F)$ and $\dim(A) = \dim(B)$.

6. Let L/F be a finite field extension with Galois group G. Let $l : G \times G \to L^\times$ be a 2-cocycle, i.e.,

$$l(\sigma, \tau) \cdot l(\sigma\tau, \rho) = \sigma(l(\tau, \rho)) \cdot l(\sigma, \tau\rho)$$

for all $\sigma, \tau, \rho \in G$. Let $A(l)$ be a vector space over L with basis e_σ for $\sigma \in G$. We make $A(l)$ into an F-algebra via the following multiplication rules:

(*i*) $e_\sigma x = \sigma(x) e_\sigma$ for all $\sigma \in G$ and $x \in L$,

(*ii*) $e_\sigma e_\tau = l(\sigma, \tau) \cdot e_{\sigma\tau}$.

Prove that $A(l)$ is a central simple F-algebra. Show that $A(l)$ does not change up to isomorphism if the cocycle l is replaced by an equivalent one: $A(l') \simeq A(l)$ if $l' \cdot l^{-1}$ is a coboundary.

In fact, the assignment $l \mapsto A(l)$ yields a group isomorphism

$$H^2(G, L^\times) \xrightarrow{\sim} \mathrm{Ker}(\mathrm{Br}(F) \to \mathrm{Br}(L)),$$

and taking these over all L we get an isomorphism

$$H^2(F, F_{\mathrm{sep}}^\times) \xrightarrow{\sim} \mathrm{Br}(F);$$

see [1, Theorem 4.4.7].

7. Let L/F be a field extension. Prove that if A is a central simple F-algebra, then $A_L := A\otimes_F L$ is a central simple L-algebra, and the assignment $A \mapsto A_L$ yields a well defined *restriction* homomorphism $\mathrm{Res} : \mathrm{Br}(F) \to \mathrm{Br}(L)$. In fact, the restriction homomorphism Res is compatible with the restriction in Galois cohomology: the diagram

$$\begin{array}{ccc} H^2(F, F_{\mathrm{sep}}^\times) & \xrightarrow{\mathrm{res}} & H^2(L, L_{\mathrm{sep}}^\times) \\ \| & & \| \\ \mathrm{Br}(F) & \xrightarrow{\mathrm{Res}} & \mathrm{Br}(L) \end{array}$$

is commutative.

8. Let V be a vector space over F_{sep}. Suppose $\Gamma = \mathrm{Gal}(F_{\mathrm{sep}}/F)$ acts on V so that $\gamma(xv) = \gamma(x)\gamma(v)$ for all $\gamma \in \Gamma$, $x \in F_{\mathrm{sep}}$ and $v \in V$. Let W be the F-subspace of Γ-invariant elements in V. Prove that the map

$$F_{\mathrm{sep}} \otimes_F W \to V \qquad x \otimes w \mapsto xw$$

is an isomorphism of vector spaces over F_{sep}.

9. Let L/F be a finite separable field extension and let A be an L-algebra. Let X be the set of all F-algebra homomorphisms $\tau : L \to F_{\mathrm{sep}}$. The Galois group $\Gamma = \mathrm{Gal}(F_{\mathrm{sep}}/F)$ acts on X by $\gamma(\tau) = \gamma \circ \tau$. For any $\tau \in X$ let A_τ be the tensor product $A \otimes_L F_{\mathrm{sep}}$ where F_{sep} is made into an L-algebra via τ, so that $ay \otimes x = a \otimes \tau(y)x$ for $a \in A$, $y \in L$ and $x \in F_{\mathrm{sep}}$. For any $\gamma \in \Gamma$ and $\tau \in X$ the map

$$\widetilde{\gamma}_\tau : A_\tau \to A_{\gamma\tau}, \quad a \otimes x \mapsto a \otimes \gamma(x)$$

is a ring isomorphism such that $\widetilde{\gamma}_\tau(xu) = \gamma(x) \cdot \widetilde{\gamma}_\tau(u)$ for $x \in F_{\mathrm{sep}}$, $u \in A_\tau$.

Consider the tensor product $B = \otimes_{\tau\in X} A_\tau$ over F_{sep}. The group Γ acts continuously on B by

$$\gamma(\otimes a_\tau) = \otimes a'_\tau \;\text{ where }\; a'_{\gamma\tau} = \widetilde{\gamma}_\tau(a_\tau).$$

Show that B is an F_{sep}-algebra such that $\gamma(xz) = \gamma(x) \cdot \gamma(z)$ for every $x \in F_{\mathrm{sep}}$ and $z \in B$. Write $N_{L/F}(A)$ for the F-subalgebra of Γ-invariant elements in B. It is called the *norm algebra of* A for the field extension L/F. Show that the natural homomorphism $F_{\mathrm{sep}} \otimes_F N_{L/F}(A) \to B$ is an isomorphism of F_{sep}-algebras and

$$\dim_F(N_{L/F}(A)) = \dim_L(A)^{[L:F]}.$$

Prove that if A is a central simple L-algebra, then $N_{L/F}(A)$ is a central simple F-algebra. Prove that the assignment $[A] \mapsto [N_{L/F}(A)]$ is a well defined group homomorphism $N_{L/F} : \mathrm{Br}(L) \to \mathrm{Br}(F)$. In fact, the norm homomorphism $N_{L/F}$ is compatible with corestriction in Galois cohomology: the diagram

$$\begin{array}{ccc} H^2(L, L_{\mathrm{sep}}^\times) & \xrightarrow{\mathrm{cor}} & H^2(F, F_{\mathrm{sep}}^\times) \\ \| & & \| \\ \mathrm{Br}(L) & \xrightarrow{N_{L/F}} & \mathrm{Br}(F) \end{array}$$

is commutative.

All fields below are of characteristic different from 2.

10. Using Hilbert's Theorem 90 prove that there a canonical isomorphism

$$h_F : H^2(F, \mathbb{Z}/2\mathbb{Z}) \xrightarrow{\sim} \mathrm{Br}(F)[2].$$

11. Let $Q = (a, b)$ be the quaternion algebra over F; see [1, Chapter 1]. Prove that Q is a central simple algebra over F. Prove that the following are equivalent:
(i) $Q \simeq M_2(F)$;
(ii) The quadratic form $\langle a, b, -ab\rangle$ is isotropic;
(iii) The 2-fold Pfister form $\langle\langle a, b\rangle\rangle$ is hyperbolic.

12. Let $a, b \in F^\times$. Prove that

$$(\chi_a \cup \chi_b)(\sigma, \tau) = \chi_a(\sigma)\chi_b(\tau)$$

in $\mathbb{Z}/2\mathbb{Z}$ and $h_F(\chi_a \cup \chi_b)$ in $\mathrm{Br}(F)[2]$ coincides with the class of the quaternion algebra (a, b).

13. Prove that $(a, b) + (a, c) = (a, bc)$ in $\mathrm{Br}(F)$.

14. Prove that the endomorphism σ of the quaternion algebra Q over a field F of characteristic different from 2 given by $\sigma(x+yi+zj+tk) = \sigma(x-yi-zj-tk)$ is an involution of Q, i.e., $\sigma(uv) = \sigma(v)\sigma(u)$ for all $u, v \in Q$ and $\sigma \circ \sigma = \mathrm{id}_Q$. Deduce that $Q^{op} \simeq Q$ and $2[Q] = 0$ in $\mathrm{Br}(F)$.

15. Under the assumptions of Problem 14, set

$$\mathrm{Trd}(q) := q + \sigma(q), \qquad \mathrm{Nrd}(q) := q \cdot \sigma(q)$$

for all $q \in Q$. Prove that $\mathrm{Trd}(q)$ is a linear map $Q \to F$ (the *reduced trace*) and $\mathrm{Nrd}(q)$ is a quadratic form $Q \to F$ (the *reduced norm*).

16. Prove that every element q of a quaternion algebra Q is a root of the quadratic polynomial $x^2 - \mathrm{Trd}(q)x + \mathrm{Nrd}(q)$ over F.

17. Let K/F be a quadratic field extension and let $Q = (x, y)$ a quaternion algebra over K.

(i) If both x and y belong to F, then $N_{K/F}(Q) = 0$ in $\mathrm{Br}(F)$ and $N_{K/F}(Q) \simeq M_4(F)$.

(ii) If $x \in F$, show that $N_{K/F}(Q) = (x, N_{K/F}(y))$ in $\mathrm{Br}(F)$ and $N_{K/F}(Q) \simeq M_2((x, N_{K/F}(y)))$.

(iii) If $x \in L \setminus F$ and $y \in L \setminus F$, let $a, b \in F^\times$ be so that $ax + by = 1$ (see Problem 9 in the Quadratic Forms section). Prove that

$$N_{K/F}(Q) = (a, N_{K/F}(y)) + (b, N_{K/F}(x)) \quad \text{in } \mathrm{Br}(F)$$

and that

$$N_{K/F}(Q) \simeq (a, N_{K/F}(y)) \otimes_F (b, N_{K/F}(x))$$

is a biquaternion algebra.

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