Amenability 2

May 23, 2017

Let G be a finitely generated group generated by a finite symmetric generating set S. For an element $g \in G$, we denote by $\ell_S(g)$ the word length of g with respect to S. We let $B_S(n) = \{g \in G : \ell_S(g) \leq n\}$ be the ball of radius n around the identity in the Cayley graph $Cay(G, S)$.

Recall that the *growth rate of G with respect to S* is

$$\omega_S(G) = \lim_{n \to \infty} \sqrt[n]{|B_S(n)|}$$

The group G has *exponential growth* if for some (equiv. any) finite symmetric generating set S, the growth rate $\omega_S(G) > 1$. Otherwise, G has *subexponential growth.*

Exercise 0.1. Prove that a finitely generated group of subexponential growth is amenable.

The next exercise shows that the converse to Exercise 0.1 does not hold. Let G and H be groups. Recall that the *(restricted) wreath product* $G \wr H$ is the semidirect product $\bigoplus_{h \in H} G \rtimes H$. The action of H on $\bigoplus_{h \in H} G$ is defined as follows. Let $f \in \bigoplus_{h \in H} G$. Then f can be viewed as a finitely supported function $f \colon H \to G$. Then, for $k \in H$, $(k * f)(h) = f(k^{-1}h)$. The wreath product $G \wr H$ consists of elements of the form (f, k) for $f \in \bigoplus_{h \in H} G$ and $k \in H$. The product is defined as follows.

$$(f_1, k_1)(f_2, k_2) = (f_1(k_1 * f_2), k_1 + k_2).$$

Exercise 0.2 (The lamplighter group - an amenable group of exponential growth)**.** The lamplighter group L_2 is defined to be the (restricted) wreath product $\mathbb{Z}_2 \wr \mathbb{Z}$. Show that

(a) L_2 is 2-generated.

(b) L_2 has exponential growth.

(c) L_2 is amenable.

Exercise 0.3 (The first Grigorchuk group)**.** The first Grigorchuk group Γ is a self-similar group generated by four automorphisms of the binary tree T. The automorphism a is the switch at the root $a = (1, 1)\sigma$ where σ is the non-trivial element of the symmetric group S_2. The automorphisms b, c and d are defined recursively by the formulas

$$b = (a, b); \quad c = (a, d); \quad d = (1, b).$$

Thus, for example, b acts trivially at the root and acts as a on the subtree rooted at the vertex 0 of the first level and as c on the subtree rooted at 1.

(a) Show that the subgroup of Γ generated by $\{b, c, d\}$ consists of the elements $\{1, b, c, d\}$ and is isomorphic to $\mathbb{Z}_2 \times \mathbb{Z}_2$. Use this to represent elements of Γ by reduced words in the alphabet a, b, c, d.

(b) For $k \geq 1$, consider the stabilizer $\operatorname{Stab}_\Gamma(k)$ of the vertices of the tree of level $\leq k$ and denote by H the stabilizer of the first level $\operatorname{Stab}_\Gamma(1)$, a subgroup of index 2 consisting of all words with even number of a's. Show that H is generated by $\{b, c, d, aba, aca, ada\}$.

(c) Consider the homomorphism

$$\varphi = (\varphi_0, \varphi_1) \colon H \to \operatorname{Aut}(T) \times \operatorname{Aut}(T)$$

which sends each element in the stabilizer of the first level to the couple of its restrictions to the subtrees rooted at the vertices of the first level. Show that $\varphi(H) \leq \Gamma \times \Gamma$ and that the projection of $\varphi(H)$ onto each of the two components is the whole Γ (i.e., that $\varphi_0(H) = \Gamma$ and $\varphi_1(H) = \Gamma$). Deduce that Γ is infinite.

(d) Use φ to show that Γ is a 2-group, that is that for any $g \in \Gamma$ there exists an integer $N \geq 0$ such that $g^{2^N} = 1$. *Hint*: Use induction on the length of g in terms of a, b, c, d. There are several cases to consider but all of them essentially boil down to the fact that the homomorphism φ is length decreasing. More precisely,

$$|\varphi_j(g)| \leq \frac{|g| + 1}{2},$$

where $|\cdot|$ stands for the word length with respect to $\{a, b, c, d\}$.

Remark 0.4. The last exercise shows that Γ is an infinite finitely generated torsion group. It is known (Chuo) that such groups cannot be elementary amenable. However Γ is amenable being of subexponential growth. Alternatively, to see that Γ is not elementary amenable one can use the fact that it has *intermediate growth* (i.e., subexponential but superpolynomial) and Chuo's result that finitely generated elementary amenable groups have either polynomial or exponential growth.