2018 IAS SUMMER COLLABORATION FINAL REPORT

SALIM ALI ALTUĞ, ARUL SHANKAR, ILA VARMA, AND KEVIN H. WILSON

During our three weeks at the IAS we focused on two questions. The first question is the following:

Problem 1. *Let D_5 denote the dihedral group of order 10, consisting of the symmetries of a regular pentagon. What is the asymptotic number of degree 5 field extensions over $\mathbb{Q}$ whose normal closure has Galois group D_5, when ordering such fields by discriminant?*

Answering this question will have applications towards the $p = 5$ case of the Cohen–Lenstra heuristics [5], as well as Malle's conjecture [7] for the group D_5.

We use a two step strategy towards answering Problem 1: first, parametrize the set of D_5-fields via group orbits on a linear space or an affine homogeneous space, and second, count these group orbits via analytic techniques. We made substantial progress towards the first step of the argument while at the IAS, which we summarize below.

Let K denote a D_5-quintic field and let $\overline{K}$ denote its Galois closure. The sextic resolvent of K is simply $\mathbb{Q} \oplus K$, and the sextic resolvent map as defined by Bhargava [3] is an alternating map $(\mathbb{Q} \oplus K)^2 \to K$, which recovers the entire multiplication table of K. Using a particular linear operator $\delta \in \mathbb{Z}[\mathrm{Gal}(\overline{K}/\mathbb{Q})]$, we give an extremely explicit description of the sextic resolvent map over any base field k containing $\mathbb{Q}(\sqrt{5})$. This allows us to describe a parametrization of D_5-quintic extensions of $\mathbb{Q}(\sqrt{5})$ in terms of certain $\mathrm{SL}_2(k) \times \mathrm{SL}_2(k)$-orbits on a $\mathrm{Gal}(\mathbb{Q}(\sqrt{5})/\mathbb{Q})$-invariant subspace of $k^2 \otimes k^2 \otimes (k^2 \oplus k^2)$. Additionally, we show that this subspace can be interpreted as a (reducible) 8-dimensional representation of $\mathrm{SL}_2(k)$, and over $\mathbb{C}$ breaks up as a direct sum of $\mathrm{Sym}^3(\mathbb{C}^2)$ and two copies of $\mathrm{Sym}^1(\mathbb{C}^2)$.

We have obtained computational and theoretical evidence that the projection of a given $\mathrm{SL}_2(k)$-orbit of the subspace to the direct summand $\mathrm{Sym}^2(k)$ should itself be enough to recover the sextic resolvent map, and hence the D_5-quintic extension of k. We are next working towards proving this. Afterwards, we will generalize these parametrizations from rational orbits to integral orbits. We will then employ and develop analytic methods to count these integral orbits and obtain asymptotics on the number of D_5-extensions of $\mathbb{Q}(\sqrt{5})$. Finally, we will be required to descend our argument to also work over $\mathbb{Q}$.

The second question is concerned with Malle's conjecture and its modification by Klüners and Türkelli [8]. Recall that this conjecture predicts the asymptotic number of degree-d number fields whose normal closure over $\mathbb{Q}$ has a fixed Galois group G, when these degree-d G-fields are ordered by discriminant. In the case of degree-d S_d-number fields, Bhargava [2] predicted a mass formula for the constant of growth. Afterwards, Kedlaya [6] described a framework for analogous mass formulae for more general Galois groups G. However, as studied by Kedlaya and Wood [9], these predictions for general G have frequently yielded the wrong asymptotics. In particular, the predicted mass formulae do not coincide with the asymptotics obtained by Cohen-Diaz y Diaz-Olivier for the family of D_4-fields ordered by discriminant [4]. Nevertheless, in previous joint work, we proved that the prediction is indeed correct for the same family ordered instead by conductor [1].

This suggests that the discriminant is not always the most natural invariant with which to order degree d G-fields when $G \neq S_d$. In our second project, we studied families of degree-d G-number fields ordered by "virtual" Artin conductors, arising from virtual characters of G. During the IAS, we also studied the following question:

Problem 2. *If one orders degree-d G-number fields by a virtual Artin conductor* $\mathrm{C}(d, G)$, *exactly when does a Hardy–Littlewood local-global principle predict the correct asymptotics?*

We give a complete conjectural answer to this question for every group G and virtual conductor $\mathrm{C}(d, G)$, simply in terms of the linear representation theory of G. In the case when G is abelian, our predictions agree with work of Wood [10], and it furthermore describes a larger class of "good" ordering invariants.

In addition, our conjectures agree with known results in the non-abelian cases when $G = S_3$, S_4, S_5, and D_4. We are currently engaged in verifying the conjecture in other known cases as well as collecting further theoretical and numerical evidence towards this conjecture.

We were very fortunate to have three dedicated weeks via the Summer Collaborators program that allowed us to make decisive progress on two important directions of our long-term collaborative project. Our time at the Institute for Advanced Study will result in multiple forthcoming publications answering Problem 1 and studying Problem 2. Overall, our stay at the IAS has contributed greatly to the future success of our joint research program, and we are grateful to the Institute for its support.

References

[1] S. A. Altug, A. Shankar, I. Varma, and K. H. Wilson. The number of quartic D_4-fields ordered by conductor. April 2017. arxiv:1704.01729.

[2] M. Bhargava. Mass formulae for extensions of local fields, and conjectures on the density of number field discriminants. *Int. Math. Res. Notices*, (17), 2007.

[3] M. Bhargava. Higher composition laws IV: The parameterization of quintic rings. *Annals of Mathematics*, 167(1):53–94, 2008.

[4] H. Cohen, F. Diaz y Diaz, and M. Olivier. Enumerating quartic dihedral extensions of $\mathbb{Q}$. *Composito Mathematica*, 133(1):65–93, 2002.

[5] H. Cohen and H. W. Lenstra, Jr. Heuristics on class groups of number fields. In *Number theory, Noordwijkerhout 1983*, volume 1068 of *Lecture Notes in Math.*, pages 33–62. 1984.

[6] K. Kedlaya. Mass formulas for local Galois representations. *Int. Math. Res. Notices*, (17), 2007. With an appendix by Daniel Gulotta.

[7] G. Malle. On the distribution of galois groups II. *Experiment. Math.*, 13(2):129–135, 2004.

[8] S. Türkelli. Connected components of Hurwitz schemes and Malle's conjecture. *Journal of Number Theory*, 155:163 – 201, 2015.

[9] M. M. Wood. Mass formulas for local Galois representations to wreath products and cross products. *Algebra & Number Theory*, 2(4):391–405, 2008.

[10] M. M. Wood. On the probabilities of local behaviors in abelian field extensions. *Compositio Mathematica*, 146(1):102–128, 2010.