Torsors over affine curves

Second problem session

Exercise 1. Let M be a locally free R–module of rank $2n \geq 2$ equipped with a regular quadratic form q. Show that, locally for the flat topology, (M, q) is hyperbolic. [Hint: One can deal first with the case of a local ring where 2 is invertible.]

Exercise 2. Let B be standard Borel R–subgroup of upper triangular matrices of $\mathrm{GL}_{2,R}$.

1. Show that the flat quotient of $\mathrm{GL}_{2,R}$ by B exists in the category of R–schemes and is isomorphic to the projective line.

2. Deduce an exact sequence of pointed sets
$$1 \to B(R) \to \mathrm{GL}_2(R) \to \mathbb{P}^1(R) \to H^1_{fppf}(R, B) \to H^1_{fppf}(R, \mathrm{GL}_2).$$

3. For R local, show that $H^1_{fppf}(R, B) = 1$ and that $H^1_{fppf}(R, \mathbb{G}_a) = 1$.

Exercise 3. Let R be a commutative ring. Let G, G' be affine group schemes over $\operatorname{Spec} R$, T be a G–torsor and $\phi : G \to G'$ be a homomorphism. We denote by $T \wedge^G G'$ the *fppf*–sheaf associated to the presheaf $S \mapsto T(S) \times_{\mathrm{Spec}(R)} G'(S)/\{(t, g') \sim (t \cdot g^{-1}, \phi(g)g')\}$.

1. Show that $T \wedge^G G'$ is a G'–torsor. We obtain a map $H^1(\phi) : H^1(X, G) \to H^1(X, G')$ of pointed sets.

2. Show that the following diagram is commutative
$$\begin{array}{ccc} H^1(X,G) & \xrightarrow{H^1(\phi)} & H^1(X,G) \\ \downarrow{\scriptstyle c_G} & & \downarrow{\scriptstyle c_{G'}} \\ \check{H}^1(X,G) & \xrightarrow{\check{H}^1(\phi)} & \check{H}^1(X,G') \end{array}$$

Exercise 4. Let R' be a finite locally free R–algebra. Let $r \geq 0$ be an integer. Let f denote the map from $\operatorname{Spec} R'$ to $\operatorname{Spec} R$.

1. Show that the R–functor $S \mapsto \mathrm{End}_{S \otimes_R R'}\Big((S \otimes_R R')^r\Big)^*$ is representable by an affine R–group scheme. We denote it by $\widetilde{G} = R_{R'/R}(\mathrm{GL}_r)$ (the Weil restriction).

2. Show that a $\mathrm{GL}_{r,R'}$–torsor is locally trivialised by an open of the form $f^{-1}(U)$ where U is an open of $\operatorname{Spec} R$.

3. Show that the category of $\widetilde{G}$–torsors is equivalent to the category of locally free R'–modules of rank r.

4. Give an interpretation of the map $H^1(R, \mathrm{GL}_r) \to H^1(R, \widetilde{G})$ and show that this map is not in general injective nor surjective.