0. Outline

Satake equivalence via convolution

work in progress, joint w/ Bezrukavnikov + ?

Goal: Describe the spherical Hecke category (derived):

$$D(G_{\mathcal{O}} \backslash Gr)$$

Conventions:

$G =$ (split) reductive $/ k$

$K = k((t)) \supset \mathcal{O} = k[[t]]$

$Gr = G_K / G_{\mathcal{O}}$

$D =$ derived category of constructible sheaves $/ E$ (coeffs)

Primarily interested in

$E = \overline{\mathbb{F}}_\ell$, $\ell \neq \operatorname{char} k$

Satake equivalence via convolution

work in progress, joint w/ Bezrukavnikov

Goal: Describe the spherical Hecke category (derived):

$$D(G_{\mathcal{O}} \backslash Gr) = D_{G_{\mathcal{O}}}(Gr)$$

Remark: char $E = 0$ is known, but the results are still interesting (also apply to D-modules, char $k = 0$).

Conventions:

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Primarily interested in

$E = \overline{\mathbb{F}}_\ell$, $\ell \neq \operatorname{char} k$

Goal: Describe $D(G_\mathcal{O}\backslash Gr)$.

Problem: Answer is hard (derived):

$$Coh\left(pt \times_{G^\vee} pt \,/\, G^\vee\right),$$

derived scheme

so proof is also hard...

But: abelian category is easy (-ish):

Mirković-Vilonen:

$$Perv(G_\mathcal{O}\backslash Gr) \simeq Rep(\check{G}).$$

Idea: Use extra structure to "bootstrap" $Perv \rightsquigarrow D$

1. Convolution

In general:

$H \supset P$ $\quad D(P\backslash H/P)$ has convolution $\star$

$P \subset H \supset Q$ $\quad D(P\backslash H/Q)$ is a $D(P\backslash H/P)$-module (and a $D(P\backslash H/P)$-$D(Q\backslash H/Q)$ bimodule).

Our setting:

$U \subset G_K \supset G_O$,

$\psi : U \longrightarrow \mathbb{G}_a$.

U = unipotent radical of Iwahori

ψ = non-degenerate character

For $X \curvearrowleft U$, we take

"$D(X/_{\psi}U) = D(X)^{U,\psi}$ = Whittaker subspace"

Ex: $G = GL_2$: $\quad U = \begin{pmatrix} 1+tO & O \\ tO & 1+tO \end{pmatrix}$,

$\psi : \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} \longmapsto a_{12} \bmod t$.

Our setting:

Spherical Hecke category: monoidal cat.

$$D(G_{\mathcal{O}}\backslash G_K/G_{\mathcal{O}}) = D(Gr)^{G_{\mathcal{O}}}$$

Unramified Whittaker category bimodule cat.

$$D(U_{\psi}\backslash G_K/G_{\mathcal{O}}) = D(Gr)^{U,\psi}$$

Bi-Whittaker category monoidal cat.

$$D(U_{\psi}\backslash G_K/_{\psi}U) = D(G_K)^{U\times U,(\psi,-\psi)}$$

2. Rigidity

Model (Boyarchenko - Drinfeld):

H = group.

$(D(H), \star)$ is rigid iff H is proper.

Proof: Dual of $\mathcal{F}$ satisfies

$$\mathrm{Hom}(\mathcal{F} \star G_1, G_2) = \mathrm{Hom}(G_1, \mathcal{F}^\vee \star G_2)$$

adjunction $\rightsquigarrow$ this would work if two versions of $\star$ coincide

$$\begin{array}{c} H \times H \\ \downarrow m \\ H \end{array} \qquad (m_* \text{ and } m_!).$$

Version: Suppose $H \supset P$ and H/P is proper. Then $(D(P\backslash H/P), \star)$ is rigid.

Monoidal category is rigid = every object is left- and right-dualizable.

Version Suppose H/P is proper.

Then for any Q_1, Q_2, consider

$$D(Q_1\backslash H/P) \times D(P\backslash H/Q_2)$$
$$\downarrow \star$$
$$D(Q_1\backslash H/Q_2)$$

Then any $\mathcal{F} \in D(Q_1\backslash H/P)$ has a right dual $\mathcal{F}^R \in D(P\backslash H/Q_1)$ (∗) and every $G \in D(P\backslash H/Q_2)$ has a left dual $\mathcal{F}^L \in D(Q_2\backslash H/P)$

Side remark:

2-category:

objects: $H \circlearrowright X$

$\mathcal{H}om(X_1, X_2) = D(X_1 \times X_2 / H)$

E.g.: $X = H/P$

$\mathcal{H}om(H/P, H/Q) = D(Q\backslash H/P)$

Then ∗ is about adjoints to 1-morphism.

Version Suppose H/P is proper.

Then for any Q_1, Q_2, consider

$$D(Q_1\backslash H/P) \times D(P\backslash H/Q_2) \xrightarrow{*} D(Q_1\backslash H/Q_2)$$

Then any $\mathcal{F} \in D(Q_1\backslash H/P)$ has a right dual $\mathcal{F}^R \in D(P\backslash H/Q_1)$ and every $G \in D(P\backslash H/Q_2)$ has a left dual $\mathcal{F}^L \in D(Q_2\backslash H/P)$

Theorem In this situation, the functor

$$D(Q_1\backslash H/P) \otimes_{D(P\backslash H/P)} D(P\backslash H/Q_2) \longrightarrow D(Q_1\backslash H/Q_2)$$

is fully faithful.

$\otimes$ of categories!

3. Putting things together

$$\mathcal{B} := D(U \backslash_\psi G_K /_\psi U)$$

$$\circlearrowright$$

$$\mathcal{W} := D(U \backslash_\psi Gr)$$

$$\circlearrowright$$

$$\mathcal{H} := D(G_{\mathcal{O}} \backslash Gr)$$

$$\text{f. faithful} \uparrow \text{Th}$$

$$\mathcal{W} \otimes_{\mathcal{B}} \mathcal{W}$$

Rem: One of $\mathcal{W}$'s should be right-to-left.

Correction: G_K/U is not proper...

but $G_K/\text{Iwahori}$ is!

So: replace

$$D(U \backslash_\psi G_K /_\psi U) \text{ with}$$

a smaller category:

sheaves that are monodromic with unipotent monodromy for T.

Then Th still applies.

$\mathcal{B} := D(U_\psi \backslash G_K /_\psi U) = \mathrm{Coh}(\tilde{\check{G}}/\check{G})_{\mathrm{Uni}}$ (work in progress, Bezrukavnikov–Riche)

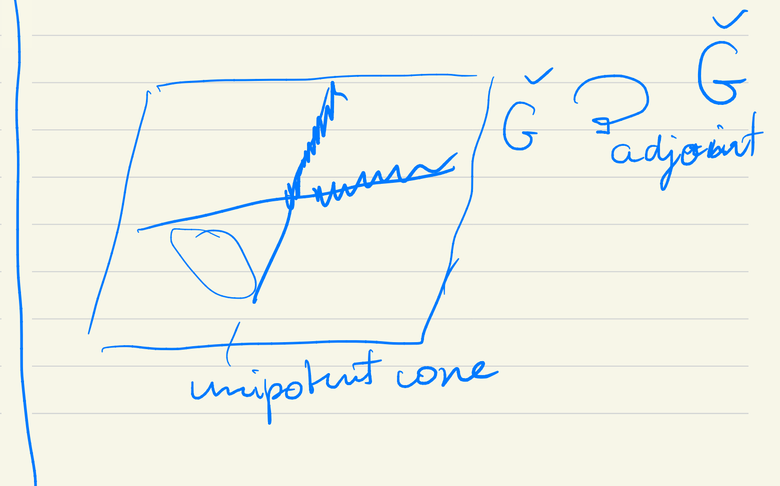
support

$\mathcal{W} := D(U_\psi \backslash \mathrm{Gr}) = \mathrm{Rep}(\check{G})$ (Casselman–Shalika of Bezrukavnikov–Gaitsgory–Mirković–Riche–Rider).

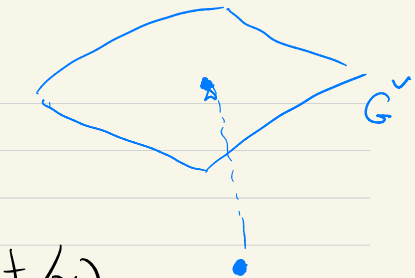
$\mathcal{H} := D(G_{\mathcal{O}} \backslash \mathrm{Gr})$

f. faithful $\uparrow$ Th

$\mathcal{W} \otimes_{\mathcal{B}} \mathcal{W}$



$$\mathcal{B} := D(U_\psi \backslash G_K / {}_\psi U) = \mathrm{Coh}(\tilde{G}/\check{G})_{\mathcal{U}nn}$$



$$\mathcal{W} := D(U_\psi \backslash Gr) = \mathrm{Rep}(\check{G}) = \mathrm{Coh}(pt/\check{G})$$

$$\mathcal{H} := D(G_O \backslash Gr) \qquad \mathrm{Coh}(pt \times_{\check{G}} pt)/\check{G}$$

f. faithful

Renormalization

$$\mathcal{W} \otimes_{\mathcal{B}} \mathcal{W} = \overset{\mathrm{Perf}}{\cancel{\mathrm{Coh}}}(pt \times_{\check{G}} pt)/\check{G}$$

$$(pt \times_{\check{G}} pt)/\check{G}$$

(derived scheme)

Summary: Hard (derived) Satake equivalence

$\Uparrow$

1. "Classical" (non-derived) equivalences (for abelian cat.).
2. Abstract nonsense (Th)
3. Calculation of image (renormalization).

A

Question How close is $W \underset{B}{\otimes} W \to \mathcal{H}$

to an equivalence?

Coherent ("B") side **Geometric ("A") side**

$M := (pt \underset{\check{G}}{\times} pt)/\check{G}$

$\mathrm{Perf}(M)$: Span of $\mathcal{O}_M \otimes V_\lambda$ $W \otimes_B W$: Span of $(-) \star (-)$

$\mathrm{Coh}(M)$: Span of $\mathcal{O}_{pt} \otimes V_\lambda$ $\mathcal{H}$: Span of IC_λ (or Δ_λ, ∇_λ)

Categories are orthogonal (in the appropriate sense):

E.g.: $\mathrm{Coh}(M) = \{ F : \mathrm{Perf}(M)^{op} \to \mathrm{Vect}^{f.dim} \mid F(A \otimes \nabla_\lambda) = 0 \text{ fixed } A, \lambda \gg 0 \}$

$\mathrm{Perf}(M) = \{ F : \mathrm{Coh}(M)^{op} \to \mathrm{Vect}^{f.dim} \mid F(A \otimes \nabla_\lambda) = 0 \text{ fixed } A, \lambda \gg 0 \}$

Remark The "intermediate" category $\mathrm{Coh}_{Nilp}(pt \underset{\check{G}}{\times} pt / \check{G})$ (corresponding to "safe" objects of Drinfeld-Gaitsgory in $\mathcal{H}$) plays no role here.

(B)

Broad picture (or: what can we see without calculations)

Answer: functor one way:

Rep$(\check{G})$ (Given: classical Satake)

commute

$$\mathcal{B} \circlearrowleft \quad \mathcal{W} = \mathrm{Rep}(\check{G}) \circlearrowright \mathcal{H}$$

(Given: Casselman–Shalika)

$$\mathrm{Perf}(\check{G}/\check{G}) \to \mathcal{B} \qquad \mathrm{Perf}(\mathrm{pt} \times_{\check{G}} \mathrm{pt} / \check{G}) \to \mathcal{H}$$

(Step 1: commutativity on the R.H.S).

(Step 2: Th)

(ignoring ind-completions)

nearby cycles/fusion