Exercises for PCMI Undergraduate Course 2024

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Some of the exercises here are taken from other resources, including May's "A Concise Course in Algebraic Topology" and Hatcher's "Algebraic Topology".

Monday, 22 July 2024: Hurewicz and cohomology

1. Use the long exact sequence in homology of pairs to argue that

$$H_q(X) = H_q(X, \emptyset) = \begin{cases} \widetilde{H}_q(X) & q \neq 0 \\ \widetilde{H}_q(X) \oplus \mathbb{Z} & q = 0 \end{cases}$$

2. Think about what's going on topologically in the Hurewicz theorem for the case $X = S^{\vee} S^1$. Here, the theorem says that $h\colon \pi_1(S^1 \vee S^1) \to \widetilde{H}_1(S^1 \vee S^1)$ is abelianization. Can you see this from the definition of the map h and the definition of CW-homology?

One can state the following relative version of the Hurewicz theorem: Let (X, A) be an $(n-1)$-connected pair ($n \geq 2$) such that A is simply connected and non-empty. Then $H_i(X, A) = 0$ for $i \leq n$ and there is an isomorphism $\pi_n(X, A) \cong H_n(X, A)$.

3. Show that this relative Hurewicz theorem in dimension n implies the non-relative one in dimension $(n-1)$ by considering the pair (CX, X).

4. Define an *acyclic* space X to be a space where $H_q(X) = \mathbb{Z}$ and the homology groups in all non-zero dimensions are trivial.
   (a) Show that the suspension of an acyclic space is also acyclic.
   (b) Prove that the suspension of an acyclic CW complex X is contractible. (Hint: you will need to use a fact we proved last week: if X is simply connected, so is ΣX.

5. Let X be an n-dimensional CW complex containing a subcomplex Y which is homotopy equivalent to an n-sphere. Use the Hurewicz theorem to prove that the map $\pi_n(Y) \to \pi_n(X)$ is injective for $n \geq 2$.

6. Show that $\widetilde{H}_q(\bigvee_i X_i) \cong \bigoplus_i \widetilde{H}_q(X_i)$ from the corresponding facts for unreduced homology and homology of pairs.

7. Let $X \vee X \to X$ be the "fold map," sending each copy of X in the wedge to X via the identity. Argue that $\widetilde{H}_q(X) \oplus \widetilde{H}_q(X) \cong \widetilde{H}_q(X \vee X) \to \widetilde{H}_q(X)$ is the addition map.

8. We explored the chain complex below on Friday. Today, we'll take $\mathrm{Hom}(-, \mathbb{Z})$ of the chain complex.

$$0 \to \mathbb{Z} \xrightarrow{0} \mathbb{Z} \xrightarrow{2} \mathbb{Z} \xrightarrow{0} \mathbb{Z} \to 0$$

   (a) For an abelian group G, define $\mathrm{Hom}(G, \mathbb{Z})$ to be the set of group homomorphism $G \to \mathbb{Z}$. Verify that $\mathrm{Hom}(G, \mathbb{Z})$ is always an abelian group.
   (b) Let $\alpha\colon G \to G'$ be a group homomorphism. Show that there is an induced homomorphism $\alpha^*\colon \mathrm{Hom}(G', \mathbb{Z}) \to \mathrm{Hom}(G, \mathbb{Z})$.

(c) Now apply $\mathrm{Hom}(-, \mathbb{Z})$ to each group and each map in the chain complex

$$0 \to \mathbb{Z} \xrightarrow{0} \mathbb{Z} \xrightarrow{2} \mathbb{Z} \xrightarrow{0} \mathbb{Z} \to 0$$

to find the associated *cochain complex.* (A "cochain complex" is just like a chain complex except that the differential d goes up in degree—$d^n : C^{n-1} \to C^n$. You can still take the homology of a cochain complex in the same way as a chain complex because you still have a $d \circ d = 0$ condition.)

(d) Calculate the homology of the cochain complex to find the cohomology of the chain complex.