Exercises for PCMI Undergraduate Course 2024

Anna Marie Bohmann and Chloe Lewis

Summer 2024

Some of the exercises here are taken from other resources, including May's "A Concise Course in Algebraic Topology" and Hatcher's "Algebraic Topology".

Friday, 19 July 2024: Chain complexes and homology

1. Compute the homology of each chain complex below:
   (a) $\ldots \to \mathbb{Z} \to \mathbb{Z} \to \ldots \to \mathbb{Z} \to \mathbb{Z} \to 0$ where every differential d_n is the zero map.
   (b) $\ldots \to \mathbb{Z} \to \mathbb{Z} \to \ldots \to \mathbb{Z} \to \mathbb{Z} \to 0$ where every differential d_n is the identity map.
   (c) $0 \to \mathbb{Z} \xrightarrow{d_1} \mathbb{Z} \to 0$ where d_1 is multiplication by 2.
   (d) $0 \to \mathbb{Z} \xrightarrow{d_2} \mathbb{Z} \oplus \mathbb{Z} \xrightarrow{d_1} \mathbb{Z} \to 0$ where $d_2(1) = (1,1)$ and $d_1(a,b) = a - b$.
   (e) $0 \to \mathbb{Z}\langle x\rangle \oplus \mathbb{Z}\langle y\rangle \xrightarrow{d_2} \mathbb{Z}\langle a\rangle \oplus \mathbb{Z}\langle b\rangle \oplus \mathbb{Z}\langle c\rangle \xrightarrow{d_1} \mathbb{Z} \to 0$ where $d_2(x) = d_2(y) = a + b - c$ and $d_1(a) = d_1(b) = d_1(c) = 0$.
2. For a CW-complex X, the map $d_n : C_n(X) \to C_{n-1}(X)$ in the cellular chain complex of X can also be described in terms of "degree."
   (a) Show that a group homomorphism $\phi : \mathbb{Z} \to \mathbb{Z}$ is entirely determined by $\phi(1)$.
   (b) Show that the (unbased) homotopy class of a map $f : S^n \to S^n$ is entirely determined by $f_*(\text{id})$, where $f_* : \pi_n(S^n) \to \pi_n(S^n)$. This integer is called the *degree* of the map f.
   (c) Observe that a choice of k-cell in X^k comes with a map $S^{k-1} \to X^{k-1}$ and a surjection $X^k/X^{k-1} \to S^k$.
   (d) Given an n-cell e^n_j with map $S^{n-1} \to X^{n-1}$ and an $n-1$ cell e^{n-1}_i with surjection $\pi_i : X^{n-1}/X^{n-2} \to S^{n-1}$, we have a composite
   $$S^{n-1} \to X^{n-1} \to X^{n-1}/X^{n-2} \xrightarrow{\pi_i} S^{n-1}.$$
   Let the degree of this map be $a_{ji} \in \mathbb{Z}$. Try to convince yourself that the map $d_n : C_n(X) \to C_{n-1}(X)$ can be specified by sending the generator of $C_n(X)$ corresponding to e^n_j to the sum $\sum_i a_{ji}[i]$, where $[i]$ is the generator of $C_{n-1}(X)$ corresponding to e^{n-1}_i.
3. Using either the description of d_n from the previous question or the description from lecture, calculate $H_*(S^2)$ where we give S^2 the CW complex structure with two 0-cells, two 1-cells, and two 2-cells.
4. Calculate the homology of $\mathbb{R}P^2$ using the CW complex structure you found yesterday.
5. A map of chain complexes $C_* \to D_*$ consists of maps $f_q : C_q \to D_q$ for each $q \in \mathbb{Z}$ such that this diagram commutes
$$\begin{array}{ccc} C_q & \xrightarrow{f_q} & D_q \\ \downarrow{\scriptstyle d_q} & & \downarrow{\scriptstyle d_q} \\ C_{q-1} & \xrightarrow{f_{q-1}} & D_{q-1} \end{array}$$
   (One often writes $f \circ d = d \circ f$, leaving the subscripts implicit.) Show that such a map induces a map $f_* : H_n(C_*) \to H_n(D_*)$ on homology in each degree.