Exercises for PCMI Undergraduate Course 2024

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Some of the exercises here are taken from other resources, including May's "A Concise Course in Algebraic Topology" and Hatcher's "Algebraic Topology".

Monday, 15 July 2024: Long Exact Sequences in Homotopy

1. Use the long exact sequence in homotopy to show that $\pi_1(\mathbb{R}P^2) \cong \mathbb{Z}/2\mathbb{Z}$ and $\pi_n(\mathbb{R}P^2) \cong \pi_n(S^2)$ for $n \geq 2$. Now find a similar identification for $\pi_n(\mathbb{R}P^m)$. What happens when $m = 1$?

2. We used the fact that $\pi_n(\Omega X) \cong \pi_{n+1}(X)$ to construct the long exact sequence of a fibration. However, as an exercise with using this long exact sequence, apply it to the fibration $\Omega X \to PX \to X$ to observe this isomorphism here.

3. Show that for $n \geq 2$,
$$\pi_n(X \vee Y) \cong \pi_n(X) \oplus \pi_n(Y) \oplus \pi_n(X \times Y, X \wedge Y).$$

When using the long exact sequence associated to a pair (X, A), you may find the following identification useful. Set J^n to be the subset of ∂I^n given by $\partial I^{n-1} \times I \cup I^{n-1} \times \{0\}$ (and set $J^0 = \{0\}$). Then we can write $\pi_n(X, A, *) = [(I^n, \partial I^n, J^n), (X, A, *)$, where this means maps and homotopies taking I^n to X, ∂I^n to A and J^n to $*$. (These are known as "maps of triples.") You can then understand the map $\partial \colon \pi_n(X, A) \to \pi_{n-1}(A)$ as just restriction to $I^{n-1} \times \{1\}$. It really helps to draw a picture for $n = 2$ here!

4. Go back to last week's exercises and tackle any ones you didn't get a chance to think about.