Exercises for PCMI Undergraduate Course 2024

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Some of the exercises here are taken from other resources, including May's "A Concise Course in Algebraic Topology" and Hatcher's "Algebraic Topology".

Friday, 12 July 2024: Fibrations and covers

1. For $n \geq 1$, define real projective space $\mathbb{R}P^n$ to be the quotient of S^n given by identifying each point with its antipode. That is, if $S^n = \{x \in \mathbb{R}^{n+1} \mid |x| =\}$, then $\mathbb{R}P^n$ is given by identifying x with $-x$. Show that $\mathbb{R}P^1$ is homeomorphic to S^1. For $n \geq 2$, $\mathbb{R}P^n \not\cong S^n$ as we'll see below. Why doesn't your argument work for $n \geq 2$?

2. Suppose $p\colon E \to B$ is a fibration and let $F_b = p^{-1}(b)$ be the fiber over the point $b \in B$. Suppose we have a path $\beta\colon I \to B$ so that $\beta(0) = b$ and $\beta(1) = b'$. By definition, there is a lift $\widetilde{\beta}$ that fits into the following diagram:

$$\begin{array}{ccccc} F_b \times \{0\} & & \xrightarrow{\quad i_b \quad} & & E \\ \downarrow & & \overset{\widetilde{\beta}}{\nearrow} & & \downarrow \\ F_b \times I & \xrightarrow{\pi_2} & I & \xrightarrow{\beta} & B \end{array}$$

   (a) Use this lift to define a map $\tau[\beta]\colon F_b \to F_{b'}$, where $F_{b'}$ is the fiber over b'. This allows us to compare F_b and $F_{b'}$.

   (b) The definition of a fibration also allows us to show that if the path β is equivalent to another path α (via a based homotopy), then $\tau[\alpha]$ is homotopic to $\tau[\beta]$. Show that this means $\tau[\beta]$ is a a homotopy equivalence with homotopy inverse $\tau[\beta^{-1}]$. Hint: if c_b is the constant path, what is $\tau[c_b]$?

   (c) (Challenge!) See if you can show the assertion in the previous problem: that the homotopy class of $\tau[\beta]$ only depends on the equivalence class of β.

3. Let $f\colon X \to Y$ be any map. Let Y^I denote (unbased) maps $I \to Y$. Define $Nf = \{(x, \chi) \in X \times Y^I \mid \chi(1) = f(x)\}$—that is, the space of pairs of a point in X and a path in Y that ends at $f(x)$. Let $\nu\colon X \to Nf$ be given by $\nu(x) = (x, c_{f(x)})$ and $\rho\colon Nf \to Y$ be given by $\rho(x, \chi) = \chi(1)$. Show that $f = \rho \circ \nu$.

4. Let $p\colon X \to Y$ be a map and let Np be as in the previous question. Recall that the definition of p begin a fibration is that whenever we have maps f and h making the solid diagram commute

$$\begin{array}{ccc} Z & \xrightarrow{f} & X \\ \downarrow & \overset{\tilde{h}}{\nearrow} & \downarrow \\ Z \times I & \xrightarrow{h} & Y \end{array}$$

   there is a lift $\tilde{h}$. Show that f is a fibration iff there is a map $s\colon Np \to X^I$ so that $s(x, \chi)(0) = x$ and $p \circ s(x, \chi) = \chi$. Hint: start by unpacking what it means for s to land in X^I so that you can make sense of the problem statement. Then note that we can identify the lift we're looking for with a map $Z \to X^I$.

5. A map $p\colon E \to B$ is a *cover* if it is surjective and each for each point $b \in B$, we can find a neighborhood V of b so that $p^{-1}(V)$ is an open set in E that is disjoint union sets each of which is homeomorphic to V when we apply p. For example, this is the case for our map $f\colon \mathbb{R} \to S^1$ from Monday's exercises.
   (a) For a point $b \in B$ and a point $e \in p^{-1}(b)$, show that a path $q\colon I \to B$ with $q(0) = b$ lifts uniquely to a path $I \to E$ starting at e. Moreover, equivalent paths lift to equivalent paths.
   (b) Use this property to argue that a cover $E \to B$ is an example of a fibration.
6. Show that $S^2 \to \mathbb{R}P^2$ is a cover.