Exercises for PCMI Undergraduate Course 2024

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Some of the exercises here are taken from other resources, including May's "A Concise Course in Algebraic Topology" and Hatcher's "Algebraic Topology".

Thursday, 11 July 2024: Pointed maps and higher homotopy groups

1. How does changing the basepoint affect $\pi_n(X)$?
2. Argue that $\pi_0(X)$ is exactly the set of path components of X. This set has a natural basepoint (how?) but no natural group structure. Why don't our constructions of products on higher homotopy groups give a group structure when $n = 0$?
3. Let I^n be the n-dimensional solid unit cube, and let ∂I^n denote its boundary. Think about why $I^n/\partial I^n \cong S^n$. Hint: Think about the quotient of the closed unit ball D^n by its boundary $\partial D^n = S^{n-1}$.
4. We argued using a picture that $\pi_n(X)$ is always Abelian for $n \geq 2$. Why doesn't our pictorial argument apply to show that $\pi_1(X)$ is Abelian?
5. Return to Question 7 from Monday, about calculating $\pi_1(S^1)$. Can you use these ideas to prove that $\pi_n(S^1) = 0$ for $n > 1$?
6. A useful lemma about functions: Suppose $f\colon A \to B$ and $g\colon B \to C$ are functions between sets and suppose that $g \circ f$ is an isomorphism/bijection. Show that f is injective and g is surjective.

The *reduced suspension* of a based space X is the quotient of the suspension SX by the line $x \times I$ where $x \in X$ is the basepoint. We denote this by ΣX.

7. Explain why $\Sigma S^n = S^{n+1}$.
8. Describe the reduced suspension of a wedge of two circles.

The *loop space* of a based space X is the collection of loops at the basepoint,

$$\Omega X = \{\gamma\colon I \to X \mid \gamma(0) = \gamma(1) = x\}.$$

Take a minute to think about why this is a different notion than the fundamental group.

9. Let X and Y be based spaces. Show that there is an isomorphism of sets,

$$[\Sigma X, Y] \cong [X, \Omega Y].$$

   Hint: Start by thinking about maps rather than based homotopy classes of maps

10. Prove that $[\Sigma X, Y]$ is a group.