Exercises for PCMI Undergraduate Course 2024

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Some of the exercises here are taken from other resources, including May's "A Concise Course in Algebraic Topology" and Hatcher's "Algebraic Topology".

Tuesday, 9 July 2024: More about the fundamental group

1. Recall that for a path a from x to y in X, we have a basechange homomorphism $\gamma[a]\colon \pi_1(X,x) \to \pi_1(X,y)$. Show that $\pi_1(X,x)$ is Abelian iff all the basechange homomorphisms depend only on the endpoints of paths, not the choice of path itself.
2. Find an example of a space where taking a different basepoint gives a different fundamental group.

The *free product* $G * H$ of groups G and H is the set of sequences (or "words") in the elements of G and H that are either

- empty
- just one element of either G or H
- alternating, in the sense that they start with an element of either G or H and then alternate elements from each of G and H, e.g. $g_1h_1g_2h_2 \cdots g_nh_n$

The product is given by concatenating words and then multiplying elements of the same group until you get to one of the above forms.

3. Show that the free product of $\mathbb{Z}$ with itself n times is the free group on n generators.
4. Show that the free product of $\mathbb{Z}/2$ with itself is infinite.

Van Kampen's theorem says the following: Let X be the union of two (path connected) open sets A and B such that $x \in A \cap B$ and $A \cap B$ is also path connected and contractible. Then $\pi_1(X,x) \cong \pi_1(A,x) * \pi_1(B,x)$.

5. Use this theorem to compute the fundamental group of a wedge of two circles.

Let G, H and K be groups and suppose we have homomorphisms $f_1\colon K \to G$ and $f_2\colon K \to H$. The *amalgamated free product* $G *_K H$ of G and H over K is quotient $(G * H)/N$ where the normalizer of the set of elements of the form $f_1(k)f_2(k)^{-1}$ for $k \in K$.

Van Kampen's Theorem, version 2: Suppose $X = A \cup B$ where A and B are path connected open subsets of X where $A \cap B$ is path connected and $x \in A \cap B$. Then $\pi_1(X,x) = \pi_1(A,x) *_{\pi_1(A\cap B,x)} \pi_1(B,x)$.

6. Use this version of van Kampen's theorem to calculate $\pi_1(S^2)$ (with any basepoint).
7. Calculate the fundamental group of a wedge of S^2's.
8. What is the fundamental group of the torus? What about many-holed tori?
9. We showed that D^2 is contractible and so $\pi_1(D^2) = 0$. In a previous exercise, you showed that $\pi_1(S^2) = 0$. Do you think S^2 is also contractible? Why or why not?