Exercises for PCMI Undergraduate Course 2024

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Some of the exercises here are taken from other resources, including May's "A Concise Course in Algebraic Topology" and Hatcher's "Algebraic Topology".

Friday, 26 July 2024: Wrapping up

1. Recall the Real projective space $\mathbb{R}P^n$, which we can think of as the quotient of S^n by the antipodal action.

   (a) Give a CW structure on $\mathbb{R}P^n$ and use it to determine the fiber of the fibration (in fact, the covering map) $S^n \to \mathbb{R}P^n$.

   Your CW structure should show that there is an inclusion $\mathbb{R}P^n \hookrightarrow \mathbb{R}P^{n+1}$ for each n. We can thus consider the colimit of inclusions of real projective spaces and define $\mathbb{R}P^\infty = \operatorname{colim}_n \mathbb{R}P^n$. Similarly, define $S^\infty = \operatorname{colim}_n S^n$.

   (b) Determine a CW structure on $\mathbb{R}P^\infty$ and S^∞.

   (c) Prove that $S^\infty \simeq *$.

2. Let $\mathbb{C}P^2$ be the complex projective plane (the space of complex lines through the origin in $\mathbb{C}^3$). We can put a CW structure on this space consisting of one 0-cell, one 2-cell, and one 4-cell. Let's compare $\mathbb{C}P^2$ with $S^2 \vee S^4$.

   (a) Calculate $\pi_1(\mathbb{C}P^2)$ - cellular approximation might be useful here. Compare this with the fundamental group of $S^2 \vee S^4$.

   (b) Compute the homology of $\mathbb{C}P^2$ using a chain complex. Compare this with the homology of $S^2 \vee S^4$.

   The *universal coefficients theorem* gives us one way to relate the homology and cohomology of a space X. In particular, it says that we get a short exact sequence for each q,

   $$0 \to \operatorname{Ext}(H_{q-1}(X), \mathbb{Z}) \to H^q(X) \to \operatorname{Hom}(H_q(X), \mathbb{Z}) \to 0.$$

   Here are two helpful facts about Ext: $\operatorname{Ext}(0, \mathbb{Z}) = 0$ and $\operatorname{Ext}(\mathbb{Z}, \mathbb{Z}) = 0$.

   (c) Use the universal coefficients theorem to calculate the cohomology of $\mathbb{C}P^2$. Compare this with the cohomology of $S^2 \vee S^4$.

   So are these two spaces the homotopy equivalent? Recall that one reason cohomology is a useful invariant is that it comes equipped with a ring structure. In the cohomology ring of $\mathbb{C}P^2$, this structure is given by $H^*(\mathbb{C}P^2) \cong \mathbb{Z}[x]/x^3$ where the generator x has degree 2.

   (d) The product on $H^*(S^2 \vee S^4)$ is of the form $H^p(S^2 \vee S^4) \times H^q(S^2 \vee S^4) \to H^{p+q}(S^2 \vee S^4)$. There is only one choice of p and q that could make this product non-zero. What is it?

   (e) Are the ring structures on $H^*(\mathbb{C}P^2)$ and $H^*(S^2 \vee S^4)$ isomorphic? What does this tell you about the spaces $\mathbb{C}P^2$ and $S^2 \vee S^4$?