Exercises for PCMI Undergraduate Course 2024

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Some of the exercises here are taken from other resources, including May's "A Concise Course in Algebraic Topology" and Hatcher's "Algebraic Topology".

Thursday, 25 July 2024: Generalized cohomology theories

1. One way to formulate the exactness axiom for a reduced cohomology theory on all (nicely) based spaces is as follows: if $g\colon X \to Y$ is a based map and Cg is the mapping cone $Y \cup_g CX$, then the sequence of maps $X \to Y \to Cg$ induces an exact sequence:

$$\widetilde{E}^q(Cg) \to \widetilde{E}^q(Y) \to \widetilde{E}^q(X)$$

   Show if there's a space Z such that $\widetilde{E}^q(X) = [X, Z]$, then this axiom must hold.

2. For any based space Z, find a "multiplication" map $\Omega Z \times \Omega Z \to \Omega Z$ that is associative up to homotopy. Now let $Z = \Omega Y$. Show that your multiplication $\Omega^2 Y \times \Omega^2 Y \to \Omega^2 Y$ is commutative up to homotopy. (Hint: think about how we defined multiplication on π_1 and how we showed π_2 is commutative.)

3. Use the previous question to argue that if $\{E_n\}$ is a spectrum, then $[X, E_n]$ is an abelian group for all n.

4. Suppose $A_1 \xrightarrow{f_1} A_2 \xrightarrow{f_2} A_3 \xrightarrow{f_3} \cdots$ is a sequence of homomorphisms of abelian groups. Define $A = \amalg A_i/\sim$, where $\sim$ is the equivalence relation given by identifying $x \in A_i$ with $f_i(x) \in A_{i+1}$ for each i.
   (a) Observe that there is a homomorphism $r_i\colon A_i \to A$ for each i.
   (b) Suppose we have another abelian group B and homomorphisms $A_i \xrightarrow{g_i} B$ for each i and that the diagrams

$$\begin{array}{ccc} A_i & \xrightarrow{f_i} & A_{i+1} \\ & \searrow^{g_i} & \downarrow{\scriptstyle g_{i+1}} \\ & & B \end{array}$$

   commute for each i. Show that there is a single homomorphism $\phi\colon A \to B$ so that the composite $A_i \xrightarrow{r_i} A \xrightarrow{\phi} B$ is g_i.

   These properties mean A is the *colimit* of the sequence and we write $A = \operatorname{colim}_i A_i$.
   (c) If all the maps f_i are isomorphisms, what is A? What if each map f_i is an inclusion of a subgroup?

5. Let X be a space and define $T_n = \Sigma^n X$. Observe that the identity map $\Sigma T_n = \Sigma(\Sigma^n X) \to T_{n+1}$ gives a map $T_n \to \Omega T_{n+1}$. This map probably isn't a homotopy equivalence, but just having maps like this means $\{T_n\}$ is a "prespectrum." Show that $\operatorname{colim}_n \pi_{q+n}(T_n)$ is the qth stable homotopy group of X. (Here the colim of maps of abelian groups means that we should take the abelian group