Exercises for PCMI Undergraduate Course 2024

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Some of the exercises here are taken from other resources, including May's "A Concise Course in Algebraic Topology" and Hatcher's "Algebraic Topology".

Tuesday, 23 July 2024: Eilenberg–MacLane spaces and cohomology

Here is another approach to constructing Eilenberg–MacLane spaces.

1. Let $n \geq 1$ and let G be an abelian group. Construct a connected CW complex $M(G, n)$ such that

$$\widetilde{H}_q(M(G,n);\mathbb{Z}) \cong \begin{cases} G & q = n \\ 0 & q \neq n \end{cases}$$

   A space with this property is called a *Moore space*—hence the letter M!
   Hint: build $M(G, n)$ as the mapping cone Cf of a map f between wedges of spheres.

2. Let $n \geq 1$ and let G be an abelian group. Construct a connected CW complex $K(G, n)$ such that

$$\pi_q(K(G,n)) \cong \begin{cases} G & q = n \\ 0 & q \neq n \end{cases}$$

   Hint: Start with $M(G, n)$ from the previous problem, apply the Hurewicz theorem, and then add cells to kill higher homotopy groups.

3. Suppose X is any connected CW complex with the property that its only nonzero homotopy group is $\pi_n(X) = G$. Construct a homotopy equivalence $K(G, n) \to X$, where $K(G, n)$ is the Eilenberg–MacLane space you constructed in the previous problem. This shows that Eilenberg–MacLane spaces are uniquely characterized (up to homotopy equivalence) by being CW complexes with their specified homotopy groups.

4. Can you find concrete examples of a space that is a $K(\mathbb{Z}, 1)$, and $K(\mathbb{Z}/2, 1)$? What about a $K(\mathbb{Z}, 2)$?

5. Go back and tackle any questions you didn't get to yesterday or last week.

6. Along the lines of Question 8 from yesterday, can you compute the associated cochain complexes to the other examples of chain complexes from Question 1 on Friday?

7. Apply $- \otimes \mathbb{Z}/2$ to the chain complex

$$0 \to \mathbb{Z} \xrightarrow{0} \mathbb{Z} \xrightarrow{2} \mathbb{Z} \xrightarrow{0} \mathbb{Z} \to 0$$

   to get a new chain complex. What is the homology of this new chain complex? Contemplate how changing coefficient groups changes the information that homology gives.