Exercises for PCMI Undergraduate Course 2024

Anna Marie Bohmann

Summer 2024

Some of the exercises here are taken from other resources, including May's "A Concise Course in Algebraic Topology."

Monday, 8 July 2024: Homotopy equivalence and fundamental groups

1. Show that homotopy is an equivalence relation on the set of continuous maps between two spaces X and Y. That is, show:
   (a) Reflexivity: $f \simeq f$ for any $f\colon X \to Y$
   (b) Symmetry: if $f \simeq g$, then $g \simeq f$ for any $f, g\colon X \to Y$
   (c) Transitivity: if $f \simeq g$ and $g \simeq h$, then $f \simeq h$, where $f, g, h\colon X \to Y$

2. Let X be a space and let $f, g\colon I \to X$ be paths in X, where the end point of f is the starting point of g. Recall that $f \cdot g$ is the "concatenated path" defined by

$$(f \cdot g)(s) = \begin{cases} f(2s) & 0 \leq s \leq \frac{1}{2} \\ g(2s-1) & \frac{1}{2} \leq s \leq 1 \end{cases}$$

   (a) Now let $h\colon I \to X$ be a third path such that the starting point of h is the end point of g. Find an explicit homotopy showing that
   $$(f \cdot g) \cdot h \simeq f \cdot (g \cdot h).$$
   (b) Recall that f^{-1} is the path f run "backwards". If the starting point of f is $x \in X$, find an explicit homotopy showing that $f \cdot f^{-1}$ is homotopic to the constant path at x.

3. Consider the definition of $f \cdot g$ given above. How much flexibility do we have in choosing how to concatenate paths? Define a new concatenation product and decide whether or not Question 2 still works with your new definition. For example, you might choose to do the first path for a larger or smaller portion of the concatenated interval. Or you might choose to run one of the paths extra fast for a while. Does this change the homotopy class of the concatenated path?

4. Show that homotopy equivalence is an equivalence relation on topological spaces.

5. Let X be any space that is *contractible*: there is a homotopy $H\colon X \times I \to X$ from the identity on X to a constant map at a point x. Show that $\pi_1(X, x) = 0$. What are some spaces that this applies to?

6. Let $f\colon D^2 \to D^2$ be continuous and have no fixed point. Let $r\colon D^2 \to S^1$ be the function that takes x to the point on the ray from $f(x)$ to x where this ray intersects S^1. Find a concrete formula for $r(x)$ in terms of $f(x)$.

7. Fill in this sketch of the proof that $\pi_1(S^1, 1) \cong \mathbb{Z}$. We identify S^1 with the complex numbers of norm 1.

(a) For each n, define a loop f_n in S^1 by $f_n(s) = e^{2\pi ins}$. Observe this loop "wraps around the circle around itself n times"—it is the composite of the loop $I \to S^1$ identifying both endpoints of I with 1 and the nth power map on S^1. Check that $[f_m][f_n] = [f_{m+n}]$.

This allows us to define a homomorphism $i\colon \mathbb{Z} \to S^1$ by $n \mapsto [f_n]$. We must check this is an isomorphism. To do this, we lift paths to $\mathbb{R}$.

(b) Define $p\colon \mathbb{R} \to S^1$ by $p(s) = e^{2\pi is}$. This map wraps each interval $[n, n+1]$ once around the circle, starting at $1 \in S^1$. Let $\tilde{f}_n\colon \mathbb{R} \to \mathbb{R}$ be defined by $\tilde{f}_n(s) = sn$. Check that $f_n = p \circ \tilde{f}_n$. We can think of $\tilde{f}_n$ as a "lift" of f_n along the map p.

(c) Now let $f\colon I \to S^1$ be any path with $f(0) = 1$. Show that there is a unique path $\tilde{f}\colon I \to \mathbb{R}$ such that $\tilde{f}(0) = 0$ and $f = p \circ \tilde{f}$. Hint: Use the fact that the inverse image under p of a sufficiently small connected neighborhood in S^1 is just a disjoint union of copies of that neighborhood contained in intervals of the form $(r+n, r+n+1)$ for all n and for some $r \in [0, 1)$. Since I is compact, we can subdivide I into finitely many closed subintervals so that f takes each of these subintervals into one of these neighborhoods. Show that the lift on each of these subintervals is just determined by where we lifted its initial point.

(d) We can thus define a function $j\colon \pi_1(S^1, 1) \to \mathbb{Z}$ by $j[f] = \tilde{f}(1)$, the endpoint of the lifted path. Show that this integer is independent of the equivalence class of $[f]$ by arguing that a homotopy $H\colon I \times I \to S^1$ lifts uniquely to homotopy $I \times I \to \mathbb{R}$. Hint: Use the fact that $I \times I$ is compact to divide into little subsquares; construct the lift subsquare by subsquare.

(e) Observe that $j[f_n] = n$ by explicit construction. Check that $j \circ i$ is the identity on $\mathbb{Z}$. This implies that i is one-to-one and j is onto. Now check that j is also one-to-one, which shows that i and j are both isomorphisms.

8. Use the fact that $\pi_1(S^1) = \mathbb{Z}$ to prove the fundamental theorem of algebra as follows. For a map $f\colon S^1 \to S^1$, we have an induced map

$$\pi_1(S^1) \xrightarrow{f_*} \pi_1(S^1).$$

While a priori this depends on a choice of basepoint, the fact that $\pi_1(S^1)$ is Abelian means it is in fact independent of this choice. (Check this!) Now define the *degree* of f to be the integer n such that $f_*(\iota) = n\iota$, where ι is the homotopy class of the identity loop on S^1. You can also think of this as the image of $1 \in \mathbb{Z}$ after identifying $\pi_1(S^1) \cong \mathbb{Z}$.

(a) Show that if $f \simeq g$, then $\deg(f) = \deg(g)$.

(b) Show that the degree of $x \mapsto x^n$ is n. Show that any constant map has degree 0.

Now suppose that $f(x) = x^n + c_1x^{n-1} + \cdots + c_{n-1}x + c_n$ is a polynomial of degree n with complex coefficients, where $n \geq 1$. We will show f must have a root. Assume f has no root on S^1 This means that $g(x) = \frac{f(x)}{|f(x)|}$ is a well-defined continuous function from S^1 to itself and we can calculate the degree of g

(c) Assume $f(x) \neq 0$ for $|x| \leq 1$. Show that the degree of g must be zero by defining a homotopy from g to a constant map.

(d) Assume $f(x) \neq 0$ for $|x| geq1$. Show that the degree of g must be n by defining a homotopy from g to the map $x \mapsto x^n$.

Since $n \neq 0$, these assumptions can't both be true, and thus f has a zero! By induction, we can thus prove that any polynomial of degree n has n roots, counted with multiplicity.

9. Show that any continuous map $f\colon S^1 \to S^1$ of degree $\neq 1$ has a fixed point.

10. Let G be a topological group and take its identity element e as a basepoint. Define the pointwise product of loops $\alpha(t)$ and $\beta(t)$ in G by $(\alpha\beta)(t) = \alpha(t)\beta(t)$. Show that $\alpha\beta$ is equivalent to the composition of path $\beta \cdot \alpha$. Use this fact to deduce that $\pi_1(G, e)$ is Abelian.

Review questions about topological spaces.

Recall that a *topological space* is a set X together with a collection $\mathcal{T}$ of subsets of X that satisfy:

- $\emptyset$ and X are both in $\mathcal{T}$
- if $U_1, \ldots, U_n$ are in $\mathcal{T}$, then so is $\bigcap_{i=1}^n U_i$
- if $\{U_\alpha\}_{\alpha \in A}$ is a collection of elements of $\mathcal{T}$, then $\bigcup_\alpha U_\alpha$ is in $\mathcal{T}$.

Here $\mathcal{T}$ is called a *topology* on X. The elements of $\mathcal{T}$ are called *open sets* in X (with this topology).

In algebraic topology, we often consider spaces up to homotopy equivalence, which means we don't actually need to consider all the details and nuances of topological spaces. Here are some questions to help you get a sense of how the general definition relates to the more familiar example of metric spaces or of $\mathbb{R}^n$.

11. Suppose (X, d) is a metric space. Define a topology $\mathcal{T}_d$ on X by declaring that a set $U \subset X$ is open in $\mathcal{T}_d$ if either U is empty or

    - for every $x \in U$, there exists $\delta > 0$ such that the ball $B_d(x, \delta) = \{y \in X \mid d(x, y) < \delta\}$ is a subset of U.

    That is, whenever U contains a point x, it must contain a little open ball around x. Show that $\mathcal{T}_d$ satisfies the definition of a topology. This is called the *metric topology* on X. (Hint: for the intersection condition, it's useful to just think of the intersection of two balls to start with.)

    You can also think of these sets U as all the sets you build as unions of open balls in the metric—check that these are the same!

One of the key ideas in topology is that of *continuous functions*. The topological definition is as follows: a function $f\colon X \to Y$ is *continuous* if whenever $V \subset Y$ is an open set in Y, then $f^{-1}(V)$ is an open set in X.

12. Show that if (X, d_X) and (Y, d_Y) are metric spaces, a function $f\colon X \to Y$ is continuous in using this open set definition if and only if it satisfies the "ε–δ definition of continuity:" f is continuous iff for every $x \in X$ and for every $\varepsilon > 0$ there exists a $\delta > 0$ such that if $d_X(x, y) < \delta$, then $d_Y(f(x), f(y)) < \varepsilon$.