Exercises for PCMI Undergraduate Course 2024

Anna Marie Bohmann and Chloe Lewis

Summer 2024

Some of the exercises here are taken from other resources, including May's "A Concise Course in Algebraic Topology" and Hatcher's "Algebraic Topology".

Monday, 8 July 2024: Homotopy equivalence and fundamental groups

1. Show that homotopy is an equivalence relation on the set of continuous maps between two spaces X and Y. That is, show:
   (a) Reflexivity: $f \simeq f$ for any $f\colon X \to Y$
   (b) Symmetry: if $f \simeq g$, then $g \simeq f$ for any $f, g\colon X \to Y$
   (c) Transitivity: if $f \simeq g$ and $g \simeq h$, then $f \simeq h$, where $f, g, h\colon X \to Y$

2. Let X be a space and let $f, g\colon I \to X$ be paths in X, where the end point of f is the starting point of g. Recall that $f \cdot g$ is the "concatenated path" defined by

$$(f \cdot g)(s) = \begin{cases} f(2s) & 0 \le s \le \frac{1}{2} \\ g(2s-1) & \frac{1}{2} \le s \le 1 \end{cases}$$

   (a) Now let $h\colon I \to X$ be a third path such that the starting point of h is the end point of g. Find an explicit homotopy showing that
$$(f \cdot g) \cdot h \simeq f \cdot (g \cdot h).$$
   (b) Recall that f^{-1} is the path f run "backwards". If the starting point of f is $x \in X$, find an explicit homotopy showing that $f \cdot f^{-1}$ is homotopic to the constant path at x.

3. Consider the definition of $f \cdot g$ given above. How much flexibility do we have in choosing how to concatenate paths? Define a new concatenation product and decide whether or not Question 2 still works with your new definition. For example, you might choose to do the first path for a larger or smaller portion of the concatenated interval. Or you might choose to run one of the paths extra fast for a while. Does this change the homotopy class of the concatenated path?

4. Show that homotopy equivalence is an equivalence relation on topological spaces.

5. Let X be any space that is *contractible*: there is a homotopy $H\colon X \times I \to X$ from the identity on X to a constant map at a point x. Show that $\pi_1(X, x) = 0$. What are some spaces that this applies to?

6. Let $f\colon D^2 \to D^2$ be continuous and have no fixed point. Let $r\colon D^2 \to S^1$ be the function that takes x to the point on the ray from $f(x)$ to x where this ray intersects S^1. Find a concrete formula for $r(x)$ in terms of $f(x)$.

7. Fill in this sketch of the proof that $\pi_1(S^1, 1) \cong \mathbb{Z}$. We identify S^1 with the complex numbers of norm 1.

(a) For each n, define a loop f_n in S^1 by $f_n(s) = e^{2\pi ins}$. Observe this loop "wraps around the circle around itself n times"—it is the composite of the loop $I \to S^1$ identifying both endpoints of I with 1 and the nth power map on S^1. Check that $[f_m][f_n] = [f_{m+n}]$.

This allows us to define a homomorphism $i\colon \mathbb{Z} \to S^1$ by $n \mapsto [f_n]$. We must check this is an isomorphism. To do this, we lift paths to $\mathbb{R}$.

(b) Define $p\colon \mathbb{R} \to S^1$ by $p(s) = e^{2\pi is}$. This map wraps each interval $[n, n+1]$ once around the circle, starting at $1 \in S^1$. Let $\tilde{f}_n\colon \mathbb{R} \to \mathbb{R}$ be defined by $\tilde{f}_n(s) = sn$. Check that $f_n = p \circ \tilde{f}_n$. We can think of $\tilde{f}_n$ as a "lift" of f_n along the map p.

(c) Now let $f\colon I \to S^1$ be any path with $f(0) = 1$. Show that there is a unique path $\tilde{f}\colon I \to \mathbb{R}$ such that $\tilde{f}(0) = 0$ and $f = p \circ \tilde{f}$. Hint: Use the fact that the inverse image under p of a sufficiently small connected neighborhood in S^1 is just a disjoint union of copies of that neighborhood contained in intervals of the form $(r+n, r+n+1)$ for all n and for some $r \in [0,1)$. Since I is compact, we can subdivide I into finitely many closed subintervals so that f takes each of these subintervals into one of these neighborhoods. Show that the lift on each of these subintervals is just determined by where we lifted its initial point.

(d) We can thus define a function $j\colon \pi_1(S^1, 1) \to \mathbb{Z}$ by $j[f] = \tilde{f}(1)$, the endpoint of the lifted path. Show that this integer is independent of the equivalence class of $[f]$ by arguing that a homotopy $H\colon I \times I \to S^1$ lifts uniquely to homotopy $I \times I \to \mathbb{R}$. Hint: Use the fact that $I \times I$ is compact to divide into little subsquares; construct the lift subsquare by subsquare.

(e) Observe that $j[f_n] = n$ by explicit construction. Check that $j \circ i$ is the identity on $\mathbb{Z}$. This implies that i is one-to-one and j is onto. Now check that j is also one-to-one, which shows that i and j are both isomorphisms.

8. Use the fact that $\pi_1(S^1) = \mathbb{Z}$ to prove the fundamental theorem of algebra as follows. For a map $f\colon S^1 \to S^1$, we have an induced map
$$\pi_1(S^1) \xrightarrow{f_*} \pi_1(S^1).$$
While a priori this depends on a choice of basepoint, the fact that $\pi_1(S^1)$ is Abelian means it is in fact independent of this choice. (Check this!) Now define the *degree* of f to be the integer n such that $f_*(\iota) = n\iota$, where ι is the homotopy class of the identity loop on S^1. You can also think of this as the image of $1 \in \mathbb{Z}$ after identifying $\pi_1(S^1) \cong \mathbb{Z}$.

(a) Show that if $f \simeq g$, then $\deg(f) = \deg(g)$.

(b) Show that the degree of $x \mapsto x^n$ is n. Show that any constant map has degree 0.

Now suppose that $f(x) = x^n + c_1 x^{n-1} + \cdots + c_{n-1}x + c_n$ is a polynomial of degree n with complex coefficients, where $n \geq 1$. We will show f must have a root. Assume f has no root on S^1 This means that $g(x) = \frac{f(x)}{|f(x)|}$ is a well-defined continuous function from S^1 to itself and we can calculate the degree of g

(c) Assume $f(x) \neq 0$ for $|x| \leq 1$. Show that the degree of g must be zero by defining a homotopy from g to a constant map.

(d) Assume $f(x) \neq 0$ for $|x| \geq 1$. Show that the degree of g must be n by defining a homotopy from g to the map $x \mapsto x^n$.

Since $n \neq 0$, these assumptions can't both be true, and thus f has a zero! By induction, we can thus prove that any polynomial of degree n has n roots, counted with multiplicity.

9. Show that any continuous map $f\colon S^1 \to S^1$ of degree $\neq 1$ has a fixed point.

10. Let G be a topological group and take its identity element e as a basepoint. Define the pointwise product of loops $\alpha(t)$ and $\beta(t)$ in G by $(\alpha\beta)(t) = \alpha(t)\beta(t)$. Show that $\alpha\beta$ is equivalent to the composition of path $\beta \cdot \alpha$. Use this fact to deduce that $\pi_1(G, e)$ is Abelian.

Review questions about topological spaces.

Recall that a *topological space* is a set X together with a collection $\mathcal{T}$ of subsets of X that satisfy:

- $\emptyset$ and X are both in $\mathcal{T}$
- if $U_1, \ldots, U_n$ are in $\mathcal{T}$, then so is $\bigcap_{i=1}^{n} U_i$
- if $\{U_\alpha\}_{\alpha \in A}$ is a collection of elements of $\mathcal{T}$, then $\bigcup_\alpha U_\alpha$ is in $\mathcal{T}$.

Here $\mathcal{T}$ is called a *topology* on X. The elements of $\mathcal{T}$ are called *open sets* in X (with this topology).

In algebraic topology, we often consider spaces up to homotopy equivalence, which means we don't actually need to consider all the details and nuances of topological spaces. Here are some questions to help you get a sense of how the general definition relates to the more familiar example of metric spaces or of $\mathbb{R}^n$.

11. Suppose (X, d) is a metric space. Define a topology $\mathcal{T}_d$ on X by declaring that a set $U \subset X$ is open in $\mathcal{T}_d$ if either U is empty or

    - for every $x \in U$, there exists $\delta > 0$ such that the ball $B_d(x, \delta) = \{y \in X \mid d(x, y) < \delta\}$ is a subset of U.

    That is, whenever U contains a point x, it must contain a little open ball around x. Show that $\mathcal{T}_d$ satisfies the definition of a topology. This is called the *metric topology* on X. (Hint: for the intersection condition, it's useful to just think of the intersection of two balls to start with.)

    You can also think of these sets U as all the sets you build as unions of open balls in the metric—check that these are the same!

One of the key ideas in topology is that of *continuous functions*. The topological definition is as follows: a function $f\colon X \to Y$ is *continuous* if whenever $V \subset Y$ is an open set in Y, then $f^{-1}(V)$ is an open set in X.

12. Show that if (X, d_X) and (Y, d_Y) are metric spaces, a function $f\colon X \to Y$ is continuous in using this open set definition if and only if it satisfies the "ε–δ definition of continuity:" f is continuous iff for every $x \in X$ and for every $\varepsilon > 0$ there exists a $\delta > 0$ such that if $d_X(x, y) < \delta$, then $d_Y(f(x), f(y)) < \varepsilon$.

Tuesday, 9 July 2024: More about the fundamental group

1. Recall that for a path a from x to y in X, we have a basechange homomorphism $\gamma[a]\colon \pi_1(X, x) \to \pi_1(X, y)$. Show that $\pi_1(X, x)$ is Abelian iff all the basechange homomorphisms depend only on the endpoints of paths, not the choice of path itself.
2. Find an example of a space where taking a different basepoint gives a different fundamental group.

The *free product* $G * H$ of groups G and H is the set of sequences (or "words") in the elements of G and H that are either

- empty
- just one element of either G or H
- alternating, in the sense that they start with an element of either G or H and then alternate elements from each of G and H, e.g. $g_1 h_1 g_2 h_2 \cdots g_n h_n$

The product is given by concatenating words and then multiplying elements of the same group until you get to one of the above forms.

3. Show that the free product of $\mathbb{Z}$ with itself n times is the free group on n generators.

4. Show that the free product of $\mathbb{Z}/2$ with itself is infinite.

Van Kampen's theorem says the following: Let X be the union of two (path connected) open sets A and B such that $x \in A \cap B$ and $A \cap B$ is also path connected and contractible. Then $\pi_1(X, x) \cong \pi_1(A, x) * \pi_1(B, x)$.

5. Use this theorem to compute the fundamental group of a wedge of two circles.

Let G, H and K be groups and suppose we have homomorphisms $f_1 \colon K \to G$ and $f_2 \colon K \to H$. The *amalgamated free product* $G *_K H$ of G and H over K is quotient $(G * H)/N$ where the normalizer of the set of elements of the form $f_1(k) f_2(k)^{-1}$ for $k \in K$.

Van Kampen's Theorem, version 2: Suppose $X = A \cup B$ where A and B are path connected open subsets of X where $A \cap B$ is path connected and $x \in A \cap B$. Then $\pi_1(X, x) = \pi_1(A, x) *_{\pi_1(A \cap B, x)} \pi_1(B, x)$.

6. Use this version of van Kampen's theorem to calculate $\pi_1(S^2)$ (with any basepoint).
7. Calculate the fundamental group of a wedge of S^2's.
8. What is the fundamental group of the torus? What about many-holed tori?
9. We showed that D^2 is contractible and so $\pi_1(D^2) = 0$. In a previous exercise, you showed that $\pi_1(S^2) = 0$. Do you think S^2 is also contractible? Why or why not?

Thursday, 11 July 2024: Pointed maps and higher homotopy groups

1. How does changing the basepoint affect $\pi_n(X)$?
2. Argue that $\pi_0(X)$ is exactly the set of path components of X. This set has a natural basepoint (how?) but no natural group structure. Why don't our constructions of products on higher homotopy groups give a group structure when $n = 0$?
3. Let I^n be the n-dimensional solid unit cube, and let ∂I^n denote its boundary. Think about why $I^n / \partial I^n \cong S^n$. Hint: Think about the quotient of the closed unit ball D^n by its boundary $\partial D^n = S^{n-1}$.
4. We argued using a picture that $\pi_n(X)$ is always Abelian for $n \geq 2$. Why doesn't our pictorial argument apply to show that $\pi_1(X)$ is Abelian?
5. Return to Question 7 from Monday, about calculating $\pi_1(S^1)$. Can you use these ideas to prove that $\pi_n(S^1) = 0$ for $n > 1$?
6. A useful lemma about functions: Suppose $f \colon A \to B$ and $g \colon B \to C$ are functions between sets and suppose that $g \circ f$ is an isomorphism/bijection. Show that f is injective and g is surjective.

The *reduced suspension* of a based space X is the quotient of the suspension SX by the line $x \times I$ where $x \in X$ is the basepoint. We denote this by ΣX.

7. Explain why $\Sigma S^n = S^{n+1}$.
8. Describe the reduced suspension of a wedge of two circles.

The *loop space* of a based space X is the collection of loops at the basepoint,

$$\Omega X = \{\gamma \colon I \to X \mid \ \gamma(0) = \gamma(1) = x\}.$$

Take a minute to think about why this is a different notion than the fundamental group.

9. Let X and Y be based spaces. Show that there is an isomorphism of sets,

$$[\Sigma X, Y] \cong [X, \Omega Y].$$

   Hint: Start by thinking about maps rather than based homotopy classes of maps

10. Prove that $[\Sigma X, Y]$ is a group.

Friday, 12 July 2024: Fibrations and covers

1. For $n \geq 1$, define real projective space $\mathbb{R}P^n$ to be the quotient of S^n given by identifying each point with its antipode. That is, if $S^n = \{x \in \mathbb{R}^{n+1} \mid |x| =\}$, then $\mathbb{R}P^n$ is given by identifying x with $-x$. Show that $\mathbb{R}P^1$ is homeomorphic to S^1. For $n \geq 2$, $\mathbb{R}P^n \not\cong S^n$ as we'll see below. Why doesn't your argument work for $n \geq 2$?

2. Suppose $p\colon E \to B$ is a fibration and let $F_b = p^{-1}(b)$ be the fiber over the point $b \in B$. Suppose we have a path $\beta\colon I \to B$ so that $\beta(0) = b$ and $\beta(1) = b'$. By definition, there is a lift $\widetilde{\beta}$ that fits into the following diagram:

$$\begin{array}{ccccc} F_b \times \{0\} & & \xrightarrow{\quad i_b \quad} & & E \\ \downarrow & & \overset{\widetilde{\beta}}{\nearrow} & & \downarrow \\ F_b \times I & \xrightarrow{\pi_2} & I & \xrightarrow{\beta} & B \end{array}$$

   (a) Use this lift to define a map $\tau[\beta]\colon F_b \to F_{b'}$, where $F_{b'}$ is the fiber over b'. This allows us to compare F_b and $F_{b'}$.

   (b) The definition of a fibration also allows us to show that if the path β is equivalent to another path α (via a based homotopy), then $\tau[\alpha]$ is homotopic to $\tau[\beta]$. Show that this means $\tau[\beta]$ is a a homotopy equivalence with homotopy inverse $\tau[\beta^{-1}]$. Hint: if c_b is the constant path, what is $\tau[c_b]$?

   (c) (Challenge!) See if you can show the assertion in the previous problem: that the homotopy class of $\tau[\beta]$ only depends on the equivalence class of β.

3. Let $f\colon X \to Y$ be any map. Let Y^I denote (unbased) maps $I \to Y$. Define $Nf = \{(x, \chi) \in X \times Y^I \mid \chi(1) = f(x)\}$—that is, the space of pairs of a point in X and a path in Y that ends at $f(x)$. Let $\nu\colon X \to Nf$ be given by $\nu(x) = (x, c_{f(x)})$ and $\rho\colon Nf \to Y$ be given by $\rho(x, \chi) = \chi(1)$. Show that $f = \rho \circ \nu$.

4. Let $p\colon X \to Y$ be a map and let Np be as in the previous question. Recall that the definition of p begin a fibration is that whenever we have maps f and h making the solid diagram commute

$$\begin{array}{ccc} Z & \xrightarrow{f} & X \\ \downarrow & \overset{\tilde{h}}{\nearrow} & \downarrow \\ Z \times I & \xrightarrow{h} & Y \end{array}$$

   there is a lift $\tilde{h}$. Show that f is a fibration iff there is a map $s\colon Np \to X^I$ so that $s(x, \chi)(0) = x$ and $p \circ s(x, \chi) = \chi$. Hint: start by unpacking what it means for s to land in X^I so that you can make sense of the problem statement. Then note that we can identify the lift we're looking for with a map $Z \to X^I$.

5. A map $p\colon E \to B$ is a *cover* if it is surjective and for each point $b \in B$, we can find a neighborhood V of b so that $p^{-1}(V)$ is an open set in E that is disjoint union sets each of which is homeomorphic to V when we apply p. For example, this is the case for our map $f\colon \mathbb{R} \to S^1$ from Monday's exercises.

   (a) For a point $b \in B$ and a point $e \in p^{-1}(b)$, show that a path $q\colon I \to B$ with $q(0) = b$ lifts uniquely to a path $I \to E$ starting at e. Moreover, equivalent paths lift to equivalent paths.

   (b) Use this property to argue that a cover $E \to B$ is an example of a fibration.

6. Show that $S^2 \to \mathbb{R}P^2$ is a cover.

Monday, 15 July 2024: Long Exact Sequences in Homotopy

1. Use the long exact sequence in homotopy to show that $\pi_1(\mathbb{R}P^2) \cong \mathbb{Z}/2\mathbb{Z}$ and $\pi_n(\mathbb{R}P^2) \cong \pi_n(S^2)$ for $n \geq 2$. Now find a similar identification for $\pi_n(\mathbb{R}P^m)$. What happens when $m = 1$?

2. We used the fact that $\pi_n(\Omega X) \cong \pi_{n+1}(X)$ to construct the long exact sequence of a fibration. However, as an exercise with using this long exact sequence, apply it to the fibration $\Omega X \to PX \to X$ to observe this isomorphism here.

3. Show that for $n \geq 2$,

$$\pi_n(X \vee Y) \cong \pi_n(X) \oplus \pi_n(Y) \oplus \pi_{n+1}(X \times Y, X \vee Y).$$

When using the long exact sequence associated to a pair (X, A), you may find the following identification useful. Set J^n to be the subset of ∂I^n given by $\partial I^{n-1} \times I \cup I^{n-1} \times \{0\}$ (and set $J^0 = \{0\}$). Then we can write $\pi_n(X, A, *) = [(I^n, \partial I^n, J^n), (X, A, *)$, where this means maps and homotopies taking I^n to X, ∂I^n to A and J^n to $*$. (These are known as "maps of triples.") You can then understand the map $\partial\colon \pi_n(X, A) \to \pi_{n-1}(A)$ as just restriction to $I^{n-1} \times \{1\}$. It really helps to draw a picture for $n = 2$ here!

4. Go back to last week's exercises and tackle any ones you didn't get a chance to think about.

Tuesday, 16 July 2024: n-connectedness and Freudenthal suspension

Homotopy excision allows us to induce an isomorphism on homotopy from the inclusion of pairs $(A, A \cap B) \to (X, B)$, **under some assumptions about n-connectedness**.

1. Show that this inclusion does not induce an isomorphism in general by considering $X = S^2 \vee S^2$ with A being the two northern hemispheres and B the two southern hemispheres. Hint: generate two long exact sequences and recall that $\pi_2(X)$ must be abelian.

2. Prove that $\pi_n(S^3) \cong \pi_n(S^2)$ for $n \geq 3$.

3. Let X be a path-connected space and define the cone on X to be $CX = X \wedge I$ (stop and think - how is this related to the suspension of X that we talked about last week?). Use the long exact sequence for the pair (CX, X) to show that $\pi_n(CX; X, *) \cong \pi_{n-1}(X, *)$ for $n \geq 1$.

We say that a space Y is *simply connected* if Y is path connected and $\pi_1(Y) = 0$.

4. Use the Freudenthal suspension theorem to prove that if a space X is path connected, then its suspension ΣX is simply connected.

5. Consider the inclusion of S^1 into $\mathbb{R}^3$ as the unit circle in the xy-plane. A tubular neighborhood of this embedding is a solid torus with this circle as its center (say of radius $1/4$ for concreteness).
   (a) Identify what the tangent space and normal space to a point $x \in S^1$ look like here.
   (b) Think about why the tubular neighborhood can be identified with the normal bundle
   (c) Can you find two ways of framing the normal bundle (that is, choosing a basis for each normal space that varies continuously as you move around the circle)?
   (d) I claim that one of these framings gives the Hopf map $S^3 \to S^2$ under the Pontryagin–Thom construction. This is not super easy to see, but think about it!

Thursday, 18 July 2024: CW complexes

1. Find a CW complex structure on $S^q \times S^q$ that has one 0-cell, two q-cells, and one $2q$-cell. (Hint: try $S^1 \times S^1$ first!) More generally, show that if X and Y are CW complexes, so is $X \times Y$.

2. Describe a CW complex structure on $\mathbb{R}P^n$.

3. Let X be a CW complex and let X^n be its n-skeleton (i.e. X^n is all the cells of dimension $\leq n$). Show that the inclusion $X^n \hookrightarrow X$ is an n-equivalence (i.e. an isomorphism on homotopy groups of dimension $< n$ and a surjection on π_n). Hint: use an approximation theorem.

4. Use Question 1 above and Question 3 from Monday to calculate $\pi_n(S^n \vee S^n)$ for $n \geq 2$.
5. First, observe that a compact CW complex has finitely many cells. Now consider $X = \{0\} \cup \{\frac{1}{n} \mid n \in \mathbb{Z}_+\}$ as a subspace of $\mathbb{R}$. Show that there is no CW complex Y such that X is homotopy equivalent to Y. (We say "X does not have the homotopy type of a CW complex" for this property.) Why doesn't this contradict our CW approximation theorem?
6. Give a CW structure on the Klein bottle. Can you find a second one?

Friday, 19 July 2024: Chain complexes and homology

1. Compute the homology of each chain complex below:
   (a) $\ldots \to \mathbb{Z} \to \mathbb{Z} \to \ldots \to \mathbb{Z} \to \mathbb{Z} \to 0$ where every differential d_n is the zero map.
   (b) $\ldots \to \mathbb{Z} \to \mathbb{Z} \to \ldots \to \mathbb{Z} \to \mathbb{Z} \to 0$ where every differential d_n is the identity map.
   (c) $0 \to \mathbb{Z} \xrightarrow{d_1} \mathbb{Z} \to 0$ where d_1 is multiplication by 2.
   (d) $0 \to \mathbb{Z} \xrightarrow{d_2} \mathbb{Z} \oplus \mathbb{Z} \xrightarrow{d_1} \mathbb{Z} \to 0$ where $d_2(1) = (1, 1)$ and $d_1(a, b) = a - b$.
   (e) $0 \to \mathbb{Z}\langle x\rangle \oplus \mathbb{Z}\langle y\rangle \xrightarrow{d_2} \mathbb{Z}\langle a\rangle \oplus \mathbb{Z}\langle b\rangle \oplus \mathbb{Z}\langle c\rangle \xrightarrow{d_1} \mathbb{Z} \to 0$ where $d_2(x) = d_2(y) = a + b - c$ and $d_1(a) = d_1(b) = d_1(c) = 0$.
2. For a CW-complex X, the map $d_n \colon C_n(X) \to C_{n-1}(X)$ in the cellular chain complex of X can also be described in terms of "degree."
   (a) Show that a group homomorphism $\phi \colon \mathbb{Z} \to \mathbb{Z}$ is entirely determined by $\phi(1)$.
   (b) Show that the (unbased) homotopy class of a map $f \colon S^n \to S^n$ is entirely determined by $f_*(\mathrm{id})$, where $f_* \colon \pi_n(S^n) \to \pi_n(S^n)$. This integer is called the *degree* of the map f.
   (c) Observe that a choice of k-cell in X^k comes with a map $S^{k-1} \to X^{k-1}$ and a surjection $X^k/X^{k-1} \to S^k$.
   (d) Given an n-cell e^n_j with map $S^{n-1} \to X^{n-1}$ and an $n-1$ cell e^{n-1}_i with surjection $\pi_i \colon X^{n-1}/X^{n-2} \to S^{n-1}$, we have a composite
   $$S^{n-1} \to X^{n-1} \to X^{n-1}/X^{n-2} \xrightarrow{\pi_i} S^{n-1}.$$
   Let the degree of this map be $a_{ji} \in \mathbb{Z}$. Try to convince yourself that the map $d_n \colon C_n(X) \to C_{n-1}(X)$ can be specified by sending the generator of $C_n(X)$ corresponding to e^n_j to the sum $\sum_i a_{ji}[i]$, where $[i]$ is the generator of $C_{n-1}(X)$ corresponding to e^{n-1}_i.
3. Using either the description of d_n from the previous question or the description from lecture, calculate $H_*(S^2)$ where we give S^2 the CW complex structure with two 0-cells, two 1-cells, and two 2-cells.
4. Calculate the homology of $\mathbb{R}P^2$ using the CW complex structure you found yesterday.
5. A map of chain complexes $C_* \to D_*$ consists of maps $f_q \colon C_q \to D_q$ for each $q \in \mathbb{Z}$ such that this diagram commutes
   $$\begin{array}{ccc} C_q & \xrightarrow{f_q} & D_q \\ \downarrow{\scriptstyle d_q} & & \downarrow{\scriptstyle d_q} \\ C_{q-1} & \xrightarrow{f_{q-1}} & D_{q-1} \end{array}$$
   (One often writes $f \circ d = d \circ f$, leaving the subscripts implicit.) Show that such a map induces a map $f_* \colon H_n(C_*) \to H_n(D_*)$ on homology in each degree.

Monday, 22 July 2024: Hurewicz and cohomology

1. Use the long exact sequence in homology of pairs to argue that

$$H_q(X) = H_q(X, \emptyset) = \begin{cases} \widetilde{H}_q(X) & q \neq 0 \\ \widetilde{H}_q(X) \oplus \mathbb{Z} & q = 0 \end{cases}$$

2. Think about what's going on topologically in the Hurewicz theorem for the case $X = S^{\vee} S^1$. Here, the theorem says that $h\colon \pi_1(S^1 \vee S^1) \to \widetilde{H}_1(S^1 \vee S^1)$ is abelianization. Can you see this from the definition of the map h and the definition of CW-homology?

One can state the following relative version of the Hurewicz theorem: Let (X, A) be an $(n-1)$-connected pair $(n \geq 2)$ such that A is simply connected and non-empty. Then $H_i(X, A) = 0$ for $i \leq n$ and there is an isomorphism $\pi_n(X, A) \cong H_n(X, A)$.

3. Show that this relative Hurewicz theorem in dimension n implies the non-relative one in dimension $(n-1)$ by considering the pair (CX, X).

4. Define an *acyclic* space X to be a space where $H_q(X) = \mathbb{Z}$ and the homology groups in all non-zero dimensions are trivial.
   (a) Show that the suspension of an acyclic space is also acyclic.
   (b) Prove that the suspension of an acyclic CW complex X is contractible. (Hint: you will need to use a fact we proved last week: if X is simply connected, so is ΣX.

5. Let X be an n-dimensional CW complex containing a subcomplex Y which is homotopy equivalent to an n-sphere. Use the Hurewicz theorem to prove that the map $\pi_n(Y) \to \pi_n(X)$ is injective for $n \geq 2$.

6. Show that $\widetilde{H}_q(\bigvee_i X_i) \cong \bigoplus_i \widetilde{H}_q(X_i)$ from the corresponding facts for unreduced homology and homology of pairs.

7. Let $X \vee X \to X$ be the "fold map," sending each copy of X in the wedge to X via the identity. Argue that $\widetilde{H}_q(X) \oplus \widetilde{H}_q(X) \cong \widetilde{H}_q(X \vee X) \to \widetilde{H}_q(X)$ is the addition map.

8. We explored the chain complex below on Friday. Today, we'll take $\mathrm{Hom}(-, \mathbb{Z})$ of the chain complex.

$$0 \to \mathbb{Z} \xrightarrow{0} \mathbb{Z} \xrightarrow{2} \mathbb{Z} \xrightarrow{0} \mathbb{Z} \to 0$$

   (a) For an abelian group G, define $\mathrm{Hom}(G, \mathbb{Z})$ to be the set of group homomorphism $G \to \mathbb{Z}$. Verify that $\mathrm{Hom}(G, \mathbb{Z})$ is always an abelian group.
   (b) Let $\alpha\colon G \to G'$ be a group homomorphism. Show that there is an induced homomorphism $\alpha^*\colon \mathrm{Hom}(G', \mathbb{Z}) \to \mathrm{Hom}(G, \mathbb{Z})$.
   (c) Now apply $\mathrm{Hom}(-, \mathbb{Z})$ to each group and each map in the chain complex

$$0 \to \mathbb{Z} \xrightarrow{0} \mathbb{Z} \xrightarrow{2} \mathbb{Z} \xrightarrow{0} \mathbb{Z} \to 0$$

   to find the associated *cochain complex*. (A "cochain complex" is just like a chain complex except that the differential d goes up in degree—$d^n\colon C^{n-1} \to C^n$. You can still take the homology of a cochain complex in the same way as a chain complex because you still have a $d \circ d = 0$ condition.)
   (d) Calculate the homology of the cochain complex to find the cohomology of the chain complex.

Tuesday, 23 July 2024: Eilenberg–MacLane spaces and cohomology

Here is another approach to constructing Eilenberg–MacLane spaces.

1. Let $n \geq 1$ and let G be an abelian group. Construct a connected CW complex $M(G, n)$ such that

$$\widetilde{H}_q(M(G,n);\mathbb{Z}) \cong \begin{cases} G & q = n \\ 0 & q \neq n \end{cases}$$

   A space with this property is called a *Moore space*—hence the letter M!
   Hint: build $M(G, n)$ as the mapping cone Cf of a map f between wedges of spheres.

2. Let $n \geq 1$ and let G be an abelian group. Construct a connected CW complex $K(G, n)$ such that

$$\pi_q(K(G,n)) \cong \begin{cases} G & q = n \\ 0 & q \neq n \end{cases}$$

   Hint: Start with $M(G, n)$ from the previous problem, apply the Hurewicz theorem, and then add cells to kill higher homotopy groups.

3. Suppose X is any connected CW complex with the property that its only nonzero homotopy group is $\pi_n(X) = G$. Construct a homotopy equivalence $K(G, n) \to X$, where $K(G, n)$ is the Eilenberg–MacLane space you constructed in the previous problem. This shows that Eilenberg–MacLane spaces are uniquely characterized (up to homotopy equivalence) by being CW complexes with their specified homotopy groups.

4. Can you find concrete examples of a space that is a $K(\mathbb{Z}, 1)$, and $K(\mathbb{Z}/2, 1)$? What about a $K(\mathbb{Z}, 2)$?

5. Go back and tackle any questions you didn't get to yesterday or last week.

6. Along the lines of Question 8 from yesterday, can you compute the associated cochain complexes to the other examples of chain complexes from Question 1 on Friday?

7. Apply $- \otimes \mathbb{Z}/2$ to the chain complex

$$0 \to \mathbb{Z} \xrightarrow{0} \mathbb{Z} \xrightarrow{2} \mathbb{Z} \xrightarrow{0} \mathbb{Z} \to 0$$

   to get a new chain complex. What is the homology of this new chain complex? Contemplate how changing coefficient groups changes the information that homology gives.

Thursday, 25 July 2024: Generalized cohomology theories

1. One way to formulate the exactness axiom for a reduced cohomology theory on all (nicely) based spaces is as follows: if $g\colon X \to Y$ is a based map and Cg is the mapping cone $Y \cup_g CX$, then the sequence of maps $X \to Y \to Cg$ induces an exact sequence:

$$\widetilde{E}^q(Cg) \to \widetilde{E}^q(Y) \to \widetilde{E}^q(X)$$

   Show if there's a space Z such that $\widetilde{E}^q(X) = [X, Z]$, then this axiom must hold.

2. For any based space Z, find a "multiplication" map $\Omega Z \times \Omega Z \to \Omega Z$ that is associative up to homotopy. Now let $Z = \Omega Y$. Show that your multiplication $\Omega^2 Y \times \Omega^2 Y \to \Omega^2 Y$ is commutative up to homotopy. (Hint: think about how we defined multiplication on π_1 and how we showed π_2 is commutative.)

3. Use the previous question to argue that if $\{E_n\}$ is a spectrum, then $[X, E_n]$ is an abelian group for all n.

4. Suppose $A_1 \xrightarrow{f_1} A_2 \xrightarrow{f_2} A_3 \xrightarrow{f_3} \cdots$ is a sequence of homomorphisms of abelian groups. Define $A = \amalg A_i / \sim$, where $\sim$ is the equivalence relation given by identifying $x \in A_i$ with $f_i(x) \in A_{i+1}$ for each i.
   (a) Observe that there is a homomorphism $r_i \colon A_i \to A$ for each i.
   (b) Suppose we have another abelian group B and homomorphisms $A_i \xrightarrow{g_i} B$ for each i and that the diagrams

   $$\begin{array}{ccc} A_i & \xrightarrow{f_i} & A_{i+1} \\ & \searrow^{g_i} & \downarrow g_{i+1} \\ & & B \end{array}$$

   commute for each i. Show that there is a single homomorphism $\phi \colon A \to B$ so that the composite $A_i \xrightarrow{r_i} A \xrightarrow{\phi} B$ is g_i.

   These properties mean A is the *colimit* of the sequence and we write $A = \operatorname{colim}_i A_i$.
   (c) If all the maps f_i are isomorphisms, what is A? What if each map f_i is an inclusion of a subgroup?
5. Let X be a space and define $T_n = \Sigma^n X$. Observe that the identity map $\Sigma T_n = \Sigma(\Sigma^n X) \to T_{n+1}$ gives a map $T_n \to \Omega T_{n+1}$. This map probably isn't a homotopy equivalence, but just having maps like this means $\{T_n\}$ is a "prespectrum." Show that $\operatorname{colim}_n \pi_{q+n}(T_n)$ is the qth stable homotopy group of X.

Friday, 26 July 2024: Wrapping up

1. Recall the Real projective space $\mathbb{R}P^n$, which we can think of as the quotient of S^n by the antipodal action.
   (a) Give a CW structure on $\mathbb{R}P^n$ and use it to determine the fiber of the fibration (in fact, the covering map) $S^n \to \mathbb{R}P^n$.

   Your CW structure should show that there is an inclusion $\mathbb{R}P^n \hookrightarrow \mathbb{R}P^{n+1}$ for each n. We can thus consider the colimit of inclusions of real projective spaces and define $\mathbb{R}P^\infty = \operatorname{colim}_n \mathbb{R}P^n$. Similarly, define $S^\infty = \operatorname{colim}_n S^n$.
   (b) Determine a CW structure on $\mathbb{R}P^\infty$ and S^∞.
   (c) Prove that $S^\infty \simeq *$.
2. Let $\mathbb{C}P^2$ be the complex projective plane (the space of complex lines through the origin in $\mathbb{C}^3$). We can put a CW structure on this space consisting of one 0-cell, one 2-cell, and one 4-cell. Let's compare $\mathbb{C}P^2$ with $S^2 \vee S^4$.
   (a) Calculate $\pi_1(\mathbb{C}P^2)$ - cellular approximation might be useful here. Compare this with the fundamental group of $S^2 \vee S^4$.
   (b) Compute the homology of $\mathbb{C}P^2$ using a chain complex. Compare this with the homology of $S^2 \vee S^4$.

   The *universal coefficients theorem* gives us one way to relate the homology and cohomology of a space X. In particular, it says that we get a short exact sequence for each q,

   $$0 \to \operatorname{Ext}(H_{q-1}(X), \mathbb{Z}) \to H^q(X) \to \operatorname{Hom}(H_q(X), \mathbb{Z}) \to 0.$$

   Here are two helpful facts about Ext: $\operatorname{Ext}(0, \mathbb{Z}) = 0$ and $\operatorname{Ext}(\mathbb{Z}, \mathbb{Z}) = 0$.
   (c) Use the universal coefficients theorem to calculate the cohomology of $\mathbb{C}P^2$. Compare this with the cohomology of $S^2 \vee S^4$.

   So are these two spaces the homotopy equivalent? Recall that one reason cohomology is a useful invariant is that it comes equipped with a ring structure. In the cohomology ring of $\mathbb{C}P^2$, this structure is given by $H^*(\mathbb{C}P^2) \cong \mathbb{Z}[x]/x^3$ where the generator x has degree 2.

(d) The product on $H^*(S^2 \vee S^4)$ is of the form $H^p(S^2 \vee S^4) \times H^q(S^2 \vee S^4) \to H^{p+q}(S^2 \vee S^4)$. There is only one choice of p and q that could make this product non-zero. What is it?

(e) Are the ring structures on $H^*(\mathbb{C}P^2)$ and $H^*(S^2 \vee S^4)$ isomorphic? What does this tell you about the spaces $\mathbb{C}P^2$ and $S^2 \vee S^4$?