Exercises for PCMI Undergraduate Course 2024

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Some of the exercises here are taken from other resources, including May's "A Concise Course in Algebraic Topology" and Hatcher's "Algebraic Topology".

Monday, 8 July 2024: Homotopy equivalence and fundamental groups

1. Show that homotopy is an equivalence relation on the set of continuous maps between two spaces X and Y. That is, show:
   (a) Reflexivity: $f \simeq f$ for any $f\colon X \to Y$
   (b) Symmetry: if $f \simeq g$, then $g \simeq f$ for any $f, g\colon X \to Y$
   (c) Transitivity: if $f \simeq g$ and $g \simeq h$, then $f \simeq h$, where $f, g, h\colon X \to Y$

2. Let X be a space and let $f, g\colon I \to X$ be paths in X, where the end point of f is the starting point of g. Recall that $f \cdot g$ is the "concatenated path" defined by

$$(f \cdot g)(s) = \begin{cases} f(2s) & 0 \le s \le \frac{1}{2} \\ g(2s-1) & \frac{1}{2} \le s \le 1 \end{cases}$$

   (a) Now let $h\colon I \to X$ be a third path such that the starting point of h is the end point of g. Find an explicit homotopy showing that
$$(f \cdot g) \cdot h \simeq f \cdot (g \cdot h).$$
   (b) Recall that f^{-1} is the path f run "backwards". If the starting point of f is $x \in X$, find an explicit homotopy showing that $f \cdot f^{-1}$ is homotopic to the constant path at x.

3. Consider the definition of $f \cdot g$ given above. How much flexibility do we have in choosing how to concatenate paths? Define a new concatenation product and decide whether or not Question 2 still works with your new definition. For example, you might choose to do the first path for a larger or smaller portion of the concatenated interval. Or you might choose to run one of the paths extra fast for a while. Does this change the homotopy class of the concatenated path?

4. Show that homotopy equivalence is an equivalence relation on topological spaces.

5. Let X be any space that is *contractible*: there is a homotopy $H\colon X \times I \to X$ from the identity on X to a constant map at a point x. Show that $\pi_1(X, x) = 0$. What are some spaces that this applies to?

6. Let $f\colon D^2 \to D^2$ be continuous and have no fixed point. Let $r\colon D^2 \to S^1$ be the function that takes x to the point on the ray from $f(x)$ to x where this ray intersects S^1. Find a concrete formula for $r(x)$ in terms of $f(x)$.

7. Fill in this sketch of the proof that $\pi_1(S^1, 1) \cong \mathbb{Z}$. We identify S^1 with the complex numbers of norm 1.

(a) For each n, define a loop f_n in S^1 by $f_n(s) = e^{2\pi i n s}$. Observe this loop "wraps around the circle around itself n times"—it is the composite of the loop $I \to S^1$ identifying both endpoints of I with 1 and the nth power map on S^1. Check that $[f_m][f_n] = [f_{m+n}]$.

This allows us to define a homomorphism $i\colon \mathbb{Z} \to S^1$ by $n \mapsto [f_n]$. We must check this is an isomorphism. To do this, we lift paths to $\mathbb{R}$.

(b) Define $p\colon \mathbb{R} \to S^1$ by $p(s) = e^{2\pi i s}$. This map wraps each interval $[n, n+1]$ once around the circle, starting at $1 \in S^1$. Let $\tilde{f}_n\colon \mathbb{R} \to \mathbb{R}$ be defined by $\tilde{f}_n(s) = sn$. Check that $f_n = p \circ \tilde{f}_n$. We can think of $\tilde{f}_n$ as a "lift" of f_n along the map p.

(c) Now let $f\colon I \to S^1$ be any path with $f(0) = 1$. Show that there is a unique path $\tilde{f}\colon I \to \mathbb{R}$ such that $\tilde{f}(0) = 0$ and $f = p \circ \tilde{f}$. Hint: Use the fact that the inverse image under p of a sufficiently small connected neighborhood in S^1 is just a disjoint union of copies of that neighborhood contained in intervals of the form $(r+n, r+n+1)$ for all n and for some $r \in [0,1)$. Since I is compact, we can subdivide I into finitely many closed subintervals so that f takes each of these subintervals into one of these neighborhoods. Show that the lift on each of these subintervals is just determined by where we lifted its initial point.

(d) We can thus define a function $j\colon \pi_1(S^1, 1) \to \mathbb{Z}$ by $j[f] = \tilde{f}(1)$, the endpoint of the lifted path. Show that this integer is independent of the equivalence class of $[f]$ by arguing that a homotopy $H\colon I \times I \to S^1$ lifts uniquely to homotopy $I \times I \to \mathbb{R}$. Hint: Use the fact that $I \times I$ is compact to divide into little subsquares; construct the lift subsquare by subsquare.

(e) Observe that $j[f_n] = n$ by explicit construction. Check that $j \circ i$ is the identity on $\mathbb{Z}$. This implies that i is one-to-one and j is onto. Now check that j is also one-to-one, which shows that i and j are both isomorphisms.

8. Use the fact that $\pi_1(S^1) = \mathbb{Z}$ to prove the fundamental theorem of algebra as follows. For a map $f\colon S^1 \to S^1$, we have an induced map

$$\pi_1(S^1) \xrightarrow{f_*} \pi_1(S^1).$$

While a priori this depends on a choice of basepoint, the fact that $\pi_1(S^1)$ is Abelian means it is in fact independent of this choice. (Check this!) Now define the *degree* of f to be the integer n such that $f_*(\iota) = n\iota$, where ι is the homotopy class of the identity loop on S^1. You can also think of this as the image of $1 \in \mathbb{Z}$ after identifying $\pi_1(S^1) \cong \mathbb{Z}$.

(a) Show that if $f \simeq g$, then $\deg(f) = \deg(g)$.

(b) Show that the degree of $x \mapsto x^n$ is n. Show that any constant map has degree 0.

Now suppose that $f(x) = x^n + c_1 x^{n-1} + \cdots + c_{n-1}x + c_n$ is a polynomial of degree n with complex coefficients, where $n \geq 1$. We will show f must have a root. Assume f has no root on S^1 This means that $g(x) = \frac{f(x)}{|f(x)|}$ is a well-defined continuous function from S^1 to itself and we can calculate the degree of g

(c) Assume $f(x) \neq 0$ for $|x| \leq 1$. Show that the degree of g must be zero by defining a homotopy from g to a constant map.

(d) Assume $f(x) \neq 0$ for $|x| \geq 1$. Show that the degree of g must be n by defining a homotopy from g to the map $x \mapsto x^n$.

Since $n \neq 0$, these assumptions can't both be true, and thus f has a zero! By induction, we can thus prove that any polynomial of degree n has n roots, counted with multiplicity.

9. Show that any continuous map $f\colon S^1 \to S^1$ of degree $\neq 1$ has a fixed point.

10. Let G be a topological group and take its identity element e as a basepoint. Define the pointwise product of loops $\alpha(t)$ and $\beta(t)$ in G by $(\alpha\beta)(t) = \alpha(t)\beta(t)$. Show that $\alpha\beta$ is equivalent to the composition of path $\beta \cdot \alpha$. Use this fact to deduce that $\pi_1(G, e)$ is Abelian.

Review questions about topological spaces.

Recall that a *topological space* is a set X together with a collection $\mathcal{T}$ of subsets of X that satisfy:

- $\emptyset$ and X are both in $\mathcal{T}$
- if $U_1, \ldots, U_n$ are in $\mathcal{T}$, then so is $\bigcap_{i=1}^{n} U_i$
- if $\{U_\alpha\}_{\alpha \in A}$ is a collection of elements of $\mathcal{T}$, then $\bigcup_\alpha U_\alpha$ is in $\mathcal{T}$.

Here $\mathcal{T}$ is called a *topology* on X. The elements of $\mathcal{T}$ are called *open sets* in X (with this topology).

In algebraic topology, we often consider spaces up to homotopy equivalence, which means we don't actually need to consider all the details and nuances of topological spaces. Here are some questions to help you get a sense of how the general definition relates to the more familiar example of metric spaces or of $\mathbb{R}^n$.

11. Suppose (X, d) is a metric space. Define a topology $\mathcal{T}_d$ on X by declaring that a set $U \subset X$ is open in $\mathcal{T}_d$ if either U is empty or

    - for every $x \in U$, there exists $\delta > 0$ such that the ball $B_d(x, \delta) = \{y \in X \mid d(x, y) < \delta\}$ is a subset of U.

    That is, whenever U contains a point x, it must contain a little open ball around x. Show that $\mathcal{T}_d$ satisfies the definition of a topology. This is called the *metric topology* on X. (Hint: for the intersection condition, it's useful to just think of the intersection of two balls to start with.)

    You can also think of these sets U as all the sets you build as unions of open balls in the metric—check that these are the same!

One of the key ideas in topology is that of *continuous functions*. The topological definition is as follows: a function $f\colon X \to Y$ is *continuous* if whenever $V \subset Y$ is an open set in Y, then $f^{-1}(V)$ is an open set in X.

12. Show that if (X, d_X) and (Y, d_Y) are metric spaces, a function $f\colon X \to Y$ is continuous in using this open set definition if and only if it satisfies the "ε–δ definition of continuity:" f is continuous iff for every $x \in X$ and for every $\varepsilon > 0$ there exists a $\delta > 0$ such that if $d_X(x, y) < \delta$, then $d_Y(f(x), f(y)) < \varepsilon$.

Tuesday, 9 July 2024: More about the fundamental group

1. Recall that for a path a from x to y in X, we have a basechange homomorphism $\gamma[a]\colon \pi_1(X, x) \to \pi_1(X, y)$. Show that $\pi_1(X, x)$ is Abelian iff all the basechange homomorphisms depend only on the endpoints of paths, not the choice of path itself.

2. Find an example of a space where taking a different basepoint gives a different fundamental group.

The *free product* $G * H$ of groups G and H is the set of sequences (or "words") in the elements of G and H that are either

- empty
- just one element of either G or H
- alternating, in the sense that they start with an element of either G or H and then alternate elements from each of G and H, e.g. $g_1 h_1 g_2 h_2 \cdots g_n h_n$

The product is given by concatenating words and then multiplying elements of the same group until you get to one of the above forms.

3. Show that the free product of $\mathbb{Z}$ with itself n times is the free group on n generators.

4. Show that the free product of $\mathbb{Z}/2$ with itself is infinite.

Van Kampen's theorem says the following: Let X be the union of two (path connected) open sets A and B such that $x \in A \cap B$ and $A \cap B$ is also path connected and contractible. Then $\pi_1(X, x) \cong \pi_1(A, x) * \pi_1(B, x)$.

5. Use this theorem to compute the fundamental group of a wedge of two circles.

Let G, H and K be groups and suppose we have homomorphisms $f_1 \colon K \to G$ and $f_2 \colon K \to H$. The *amalgamated free product* $G *_K H$ of G and H over K is quotient $(G * H)/N$ where the normalizer of the set of elements of the form $f_1(k)f_2(k)^{-1}$ for $k \in K$.

Van Kampen's Theorem, version 2: Suppose $X = A \cup B$ where A and B are path connected open subsets of X where $A \cap B$ is path connected and $x \in A \cap B$. Then $\pi_1(X, x) = \pi_1(A, x) *_{\pi_1(A \cap B, x)} \pi_1(B, x)$.

6. Use this version of van Kampen's theorem to calculate $\pi_1(S^2)$ (with any basepoint).
7. Calculate the fundamental group of a wedge of S^2's.
8. What is the fundamental group of the torus? What about many-holed tori?
9. We showed that D^2 is contractible and so $\pi_1(D^2) = 0$. In a previous exercise, you showed that $\pi_1(S^2) = 0$. Do you think S^2 is also contractible? Why or why not?

Thursday, 11 July 2024: Pointed maps and higher homotopy groups

1. How does changing the basepoint affect $\pi_n(X)$?
2. Argue that $\pi_0(X)$ is exactly the set of path components of X. This set has a natural basepoint (how?) but no natural group structure. Why don't our constructions of products on higher homotopy groups give a group structure when $n = 0$?
3. Let I^n be the n-dimensional solid unit cube, and let ∂I^n denote its boundary. Think about why $I^n/\partial I^n \cong S^n$. Hint: Think about the quotient of the closed unit ball D^n by its boundary $\partial D^n = S^{n-1}$.
4. We argued using a picture that $\pi_n(X)$ is always Abelian for $n \geq 2$. Why doesn't our pictorial argument apply to show that $\pi_1(X)$ is Abelian?
5. Return to Question 7 from Monday, about calculating $\pi_1(S^1)$. Can you use these ideas to prove that $\pi_n(S^1) = 0$ for $n > 1$?
6. A useful lemma about functions: Suppose $f \colon A \to B$ and $g \colon B \to C$ are functions between sets and suppose that $g \circ f$ is an isomorphism/bijection. Show that f is injective and g is surjective.

The *reduced suspension* of a based space X is the quotient of the suspension SX by the line $x \times I$ where $x \in X$ is the basepoint. We denote this by ΣX.

7. Explain why $\Sigma S^n = S^{n+1}$.
8. Describe the reduced suspension of a wedge of two circles.

The *loop space* of a based space X is the collection of loops at the basepoint,

$$\Omega X = \{\gamma \colon I \to X \mid \ \gamma(0) = \gamma(1) = x\}.$$

Take a minute to think about why this is a different notion than the fundamental group.

9. Let X and Y be based spaces. Show that there is an isomorphism of sets,

$$[\Sigma X, Y] \cong [X, \Omega Y].$$

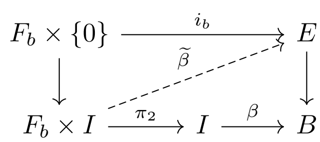
   Hint: Start by thinking about maps rather than based homotopy classes of maps

10. Prove that $[\Sigma X, Y]$ is a group.

Friday, 12 July 2024: Fibrations and covers

1. For $n \geq 1$, define real projective space $\mathbb{R}P^n$ to be the quotient of S^n given by identifying each point with its antipode. That is, if $S^n = \{x \in \mathbb{R}^{n+1} \mid |x| =\}$, then $\mathbb{R}P^n$ is given by identifying x with $-x$. Show that $\mathbb{R}P^1$ is homeomorphic to S^1. For $n \geq 2$, $\mathbb{R}P^n \not\cong S^n$ as we'll see below. Why doesn't your argument work for $n \geq 2$?

2. Suppose $p\colon E \to B$ is a fibration and let $F_b = p^{-1}(b)$ be the fiber over the point $b \in B$. Suppose we have a path $\beta\colon I \to B$ so that $\beta(0) = b$ and $\beta(1) = b'$. By definition, there is a lift $\widetilde{\beta}$ that fits into the following diagram:



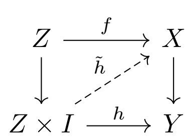
   (a) Use this lift to define a map $\tau[\beta]\colon F_b \to F_{b'}$, where $F_{b'}$ is the fiber over b'. This allows us to compare F_b and $F_{b'}$.

   (b) The definition of a fibration also allows us to show that if the path β is equivalent to another path α (via a based homotopy), then $\tau[\alpha]$ is homotopic to $\tau[\beta]$. Show that this means $\tau[\beta]$ is a a homotopy equivalence with homotopy inverse $\tau[\beta^{-1}]$. Hint: if c_b is the constant path, what is $\tau[c_b]$?

   (c) (Challenge!) See if you can show the assertion in the previous problem: that the homotopy class of $\tau[\beta]$ only depends on the equivalence class of β.

3. Let $f\colon X \to Y$ be any map. Let Y^I denote (unbased) maps $I \to Y$. Define $Nf = \{(x, \chi) \in X \times Y^I \mid \chi(1) = f(x)\}$—that is, the space of pairs of a point in X and a path in Y that ends at $f(x)$. Let $\nu\colon X \to Nf$ be given by $\nu(x) = (x, c_{f(x)})$ and $\rho\colon Nf \to Y$ be given by $\rho(x, \chi) = \chi(1)$. Show that $f = \rho \circ \nu$.

4. Let $p\colon X \to Y$ be a map and let Np be as in the previous question. Recall that the definition of p begin a fibration is that whenever we have maps f and h making the solid diagram commute



   there is a lift $\tilde{h}$. Show that f is a fibration iff there is a map $s\colon Np \to X^I$ so that $s(x, \chi)(0) = x$ and $p \circ s(x, \chi) = \chi$. Hint: start by unpacking what it means for s to land in X^I so that you can make sense of the problem statement. Then note that we can identify the lift we're looking for with a map $Z \to X^I$.

5. A map $p\colon E \to B$ is a *cover* if it is surjective and for each point $b \in B$, we can find a neighborhood V of b so that $p^{-1}(V)$ is an open set in E that is disjoint union sets each of which is homeomorphic to V when we apply p. For example, this is the case for our map $f\colon \mathbb{R} \to S^1$ from Monday's exercises.

   (a) For a point $b \in B$ and a point $e \in p^{-1}(b)$, show that a path $q\colon I \to B$ with $q(0) = b$ lifts uniquely to a path $I \to E$ starting at e. Moreover, equivalent paths lift to equivalent paths.

   (b) Use this property to argue that a cover $E \to B$ is an example of a fibration.

6. Show that $S^2 \to \mathbb{R}P^2$ is a cover.

Monday, 15 July 2024: Long Exact Sequences in Homotopy

1. Use the long exact sequence in homotopy to show that $\pi_1(\mathbb{R}P^2) \cong \mathbb{Z}/2\mathbb{Z}$ and $\pi_n(\mathbb{R}P^2) \cong \pi_n(S^2)$ for $n \geq 2$. Now find a similar identification for $\pi_n(\mathbb{R}P^m)$. What happens when $m = 1$?

2. We used the fact that $\pi_n(\Omega X) \cong \pi_{n+1}(X)$ to construct the long exact sequence of a fibration. However, as an exercise with using this long exact sequence, apply it to the fibration $\Omega X \to PX \to X$ to observe this isomorphism here.

3. Show that for $n \geq 2$,

$$\pi_n(X \vee Y) \cong \pi_n(X) \oplus \pi_n(Y) \oplus \pi_{n+1}(X \times Y, X \vee Y).$$

When using the long exact sequence associated to a pair (X, A), you may find the following identification useful. Set J^n to be the subset of ∂I^n given by $\partial I^{n-1} \times I \cup I^{n-1} \times \{0\}$ (and set $J^0 = \{0\}$). Then we can write $\pi_n(X, A, *) = [(I^n, \partial I^n, J^n), (X, A, *)$, where this means maps and homotopies taking I^n to X, ∂I^n to A and J^n to $*$. (These are known as "maps of triples.") You can then understand the map $\partial\colon \pi_n(X, A) \to \pi_{n-1}(A)$ as just restriction to $I^{n-1} \times \{1\}$. It really helps to draw a picture for $n = 2$ here!

4. Go back to last week's exercises and tackle any ones you didn't get a chance to think about.

Tuesday, 16 July 2024: n-connectedness and Freudenthal suspension

Homotopy excision allows us to induce an isomorphism on homotopy from the inclusion of pairs $(A, A \cap B) \to (X, B)$, **under some assumptions about n-connectedness**.

1. Show that this inclusion does not induce an isomorphism in general by considering $X = S^2 \vee S^2$ with A being the two northern hemispheres and B the two southern hemispheres. Hint: generate two long exact sequences and recall that $\pi_2(X)$ must be abelian.

2. Prove that $\pi_n(S^3) \cong \pi_n(S^2)$ for $n \geq 3$.

3. Let X be a path-connected space and define the cone on X to be $CX = X \wedge I$ (stop and think - how is this related to the suspension of X that we talked about last week?). Use the long exact sequence for the pair (CX, X) to show that $\pi_n(CX; X, *) \cong \pi_{n-1}(X, *)$ for $n \geq 1$.

We say that a space Y is *simply connected* if Y is path connected and $\pi_1(Y) = 0$.

4. Use the Freudenthal suspension theorem to prove that if a space X is path connected, then its suspension ΣX is simply connected.

5. Consider the inclusion of S^1 into $\mathbb{R}^3$ as the unit circle in the xy-plane. A tubular neighborhood of this embedding is a solid torus with this circle as its center (say of radius 1/4 for concreteness).
   (a) Identify what the tangent space and normal space to a point $x \in S^1$ look like here.
   (b) Think about why the tubular neighborhood can be identified with the normal bundle
   (c) Can you find two ways of framing the normal bundle (that is, choosing a basis for each normal space that varies continuously as you move around the circle)?
   (d) I claim that one of these framings gives the Hopf map $S^3 \to S^2$ under the Pontryagin–Thom construction. This is not super easy to see, but think about it!

Thursday, 18 July 2024: CW complexes

1. Find a CW complex structure on $S^q \times S^q$ that has one 0-cell, two q-cells, and one $2q$-cell. (Hint: try $S^1 \times S^1$ first!) More generally, show that if X and Y are CW complexes, so is $X \times Y$.

2. Describe a CW complex structure on $\mathbb{R}P^n$.

3. Let X be a CW complex and let X^n be its n-skeleton (i.e. X^n is all the cells of dimension $\leq n$). Show that the inclusion $X^n \hookrightarrow X$ is an n-equivalence (i.e. an isomorphism on homotopy groups of dimension $< n$ and a surjection on π_n). Hint: use an approximation theorem.

4. Use Question 1 above and Question 3 from Monday to calculate $\pi_n(S^n \vee S^n)$ for $n \geq 2$.

5. First, observe that a compact CW complex has finitely many cells. Now consider $X = \{0\} \cup \{\frac{1}{n} \mid n \in \mathbb{Z}_+\}$ as a subspace of $\mathbb{R}$. Show that there is no CW complex Y such that X is homotopy equivalent to Y. (We say "X does not have the homotopy type of a CW complex" for this property.) Why doesn't this contradict our CW approximation theorem?

6. Give a CW structure on the Klein bottle. Can you find a second one?

Friday, 19 July 2024: Chain complexes and homology

1. Compute the homology of each chain complex below:
   (a) $\ldots \to \mathbb{Z} \to \mathbb{Z} \to \ldots \to \mathbb{Z} \to \mathbb{Z} \to 0$ where every differential d_n is the zero map.
   (b) $\ldots \to \mathbb{Z} \to \mathbb{Z} \to \ldots \to \mathbb{Z} \to \mathbb{Z} \to 0$ where every differential d_n is the identity map.
   (c) $0 \to \mathbb{Z} \xrightarrow{d_1} \mathbb{Z} \to 0$ where d_1 is multiplication by 2.
   (d) $0 \to \mathbb{Z} \xrightarrow{d_2} \mathbb{Z} \oplus \mathbb{Z} \xrightarrow{d_1} \mathbb{Z} \to 0$ where $d_2(1) = (1,1)$ and $d_1(a,b) = a - b$.
   (e) $0 \to \mathbb{Z}\langle x\rangle \oplus \mathbb{Z}\langle y\rangle \xrightarrow{d_2} \mathbb{Z}\langle a\rangle \oplus \mathbb{Z}\langle b\rangle \oplus \mathbb{Z}\langle c\rangle \xrightarrow{d_1} \mathbb{Z} \to 0$ where $d_2(x) = d_2(y) = a + b - c$ and $d_1(a) = d_1(b) = d_1(c) = 0$.

2. For a CW-complex X, the map $d_n : C_n(X) \to C_{n-1}(X)$ in the cellular chain complex of X can also be described in terms of "degree."
   (a) Show that a group homomorphism $\phi \colon \mathbb{Z} \to \mathbb{Z}$ is entirely determined by $\phi(1)$.
   (b) Show that the (unbased) homotopy class of a map $f \colon S^n \to S^n$ is entirely determined by $f_*(\mathrm{id})$, where $f_* \colon \pi_n(S^n) \to \pi_n(S^n)$. This integer is called the *degree* of the map f.
   (c) Observe that a choice of k-cell in X^k comes with a map $S^{k-1} \to X^{k-1}$ and a surjection $X^k/X^{k-1} \to S^k$.
   (d) Given an n-cell e_j^n with map $S^{n-1} \to X^{n-1}$ and an $n-1$ cell e_i^{n-1} with surjection $\pi_i \colon X^{n-1}/X^{n-2} \to S^{n-1}$, we have a composite

   $$S^{n-1} \to X^{n-1} \to X^{n-1}/X^{n-2} \xrightarrow{\pi_i} S^{n-1}.$$

   Let the degree of this map be $a_{ji} \in \mathbb{Z}$. Try to convince yourself that the map $d_n : C_n(X) \to C_{n-1}(X)$ can be specified by sending the generator of $C_n(X)$ corresponding to e_j^n to the sum $\sum_i a_{ji}[i]$, where $[i]$ is the generator of $C_{n-1}(X)$ corresponding to e_i^{n-1}.

3. Using either the description of d_n from the previous question or the description from lecture, calculate $H_*(S^2)$ where we give S^2 the CW complex structure with two 0-cells, two 1-cells, and two 2-cells.

4. Calculate the homology of $\mathbb{R}P^2$ using the CW complex structure you found yesterday.

5. A map of chain complexes $C_* \to D_*$ consists of maps $f_q \colon C_q \to D_q$ for each $q \in \mathbb{Z}$ such that this diagram commutes

$$\begin{array}{ccc} C_q & \xrightarrow{f_q} & D_q \\ \downarrow{\scriptstyle d_q} & & \downarrow{\scriptstyle d_q} \\ C_{q-1} & \xrightarrow{f_{q-1}} & D_{q-1} \end{array}$$

   (One often writes $f \circ d = d \circ f$, leaving the subscripts implicit.) Show that such a map induces a map $f_* \colon H_n(C_*) \to H_n(D_*)$ on homology in each degree.