CHARACTERISTIC CLASSES IN MOTIVIC HOMOTOPY THEORY

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Abstract. These are the notes of a course given at the 2024 PCMI summer school "Motivic homotopy theory", organized by Ben Antieau, Marc Levine, Oliver Röndigs, Sasha Vishik and Kirsten Wickelgren.

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Conventions. Schemes will be Noetherian. Smooth will mean smooth of finite type. Unless stated otherwise, the Grothendieck topology on $\mathscr{S}m_S$ is the Nisnvich topology. Therefore, sheaves will mean sheaves for the Nisnevich topology. In the third course, we sometimes use the shortcut s-morphism for separated morphism of finite type.

We use the language of ∞-categories.[1] We let Cat_∞ (resp. $Cat_\infty^\otimes$) be the ∞-category of presentable ∞-categories with left adjoint ∞-functors (resp. presentable symmetric closed monoidal ∞-category

[1] One can obtain an explicit presentation of the categories here by using classical model categories on simplicial sheaves: the invective and Nisnevich-local model category structures on simplicial sheaves, and their Bousfield $\mathbb{A}^1$-localization.

with left adjoint and symmetric monoidal ∞-functors). All our ∞-categories will be presentable ∞-categories. Similarly, monoidal ∞-categories will be presentable monoidal ∞-categories. On the other hand, we mostly work in the associated homotopy category in this course.

Monoidal means symmetric monoidal. All our monoids are commutative. We denote by $\mathbb{1}_S$ the "sphere spectrum" over S *i.e.* the unit of the monoidal structure on $\mathrm{SH}(S)$.

Unless stated otherwise, spectrum means motivic spectrum, and ring spectrum means commutative motivic spectrum.

Cours 1. **Oriented spectra and Chern classes**

Introduction

The theory of *characteristic classes* of *fiber bundles* arose at exactly the same time than (singular) cohomology, in 1935. That year, Stiefel (in his PhD) and Whitney both introduced the notion of fiber bundle and some associated characteristic class.[2] Meanwhile, at the Moscow international conference on topology, Alexander and Kolmogorov independently introduced cohomology and the (soon to be called) cup-product.

The history of the subject of characteristic classes was then marked by the introduction of *Pontryagin classes*, out of the computation of the homology of real grassmanianns, by Pontryagin in 1942, and by the introduction of *Chern classes*, obtained through the determination of the cohomology of complex Grassmanians by Chern in 1946. A last event I want to mention is the course "characteristic classes" given at the University of Princeton by John Milnor in 1957.[3]

Here is a list of the characteristic classes that emerged from the works mentioned above:[4]

name	fiber b.	notation	group
Stiefel-Whitney	smooth real v.b.	w_i	$H^i(B, \mathbb{Z}/2)$
Pontryagin	smooth real v.b.	p_i	$H^{4i}(B, \mathbb{Z})$
Chern	smooth complex v.b.	c_i	$H^{2i}(B, \mathbb{Z})$

[2]The terminology "fiber bundle" is due to Stiefel, though he actually only considered smooth real vector bundles, while it was Whitney that formally introduced the so-called characteristic classes.

[3]Notes by Stasheff were available at that time. They were finally published in 1974, [MS74].

[4]Recall that Pontryagin classes are actually particular cases of Chern classes according to the formula: $p_i(V) = (-1)^i.c_{2i}(V \otimes_{\mathbb{R}} \mathbb{C})$, where $V \otimes_{\mathbb{R}} \mathbb{C}$ is the complexification of the real vector space V.

Note that fiber bundle had already appeared: first in works of E. Cartan on Lie groups and their associated homogeneous spaces, and in the work of Stiefel, slightly earlier in 1933, who was interested in constructing new 3-dimensional varieties (in view of the Poincaré conjecture). Recall that in the most general form, a fiber bundle is a map $p : E \to B$ such that there exists an open cover $W \to B$ and a W-homeomorphism: $(F \times W) \to E \times_B W$ for some space F.[5] One calls B, E, F respectively the base (space), the total space and the fiber (space) of the fiber bundle.[6]

Example 1.0.1. Here are some of the most famous examples of fiber bundles:

- *tangent bundles.* $p : TM \to M$, projection from the tangent bundle of a smooth (resp. analytic) manifold M. This is a particular case of smooth real (resp. complex analytic) vector bundles.
- *homogeneous spaces.* for G a Lie group, and $H \subset G$ a closed subgroup, $p : G \to G/H$. This is a particular case of a principal G-bundle.
- *Covering spaces.* $P \to X$. The fiber is then a *discrete* space.
- The Möbius strip T is (non trivially!) fibered over S^1, the map $T \to S^1$ being the projection.
- The Hopf fibration: $S^3 \to S^2$, with fiber S^1.

In the previous list, only vector bundles were considered. In topology, more general fiber bundles naturally appear in the so-called *obstruction theory.* They arise as morphisms in the Postnikov tower, in good cases (simple, or more generally nilpotent spaces). The attendees have already seen this theory at work in the talk of Aravind Asok: primary and secondary obstructions can be seen as characteristic classes In this course however, we will focus on algebraic vector bundles, in order to draw a picture similar to the above table.

Characteristic classes are invariant under isomorphism of fiber bundles. In particular, they can differentiate the homotopy type of the total space. However, they are far from determining this homotopy

[5]Variants arise first by working in other categories than topological spaces. In algebraic geometry, one also considers covers from various (Grothendieck) topologies: Zariski, Nisnevich, étale, fppf (mainly).

[6]Remark that such a fiber space is in particular a Hurewicz fibration, and therefore a Serre fibration. So one sometimes abusively says “fibration” for “fiber bundle”.

type (let alone the diffeomorphism type), even if one adds the homotopy type of the base.[7] In his groundbreaking 1954 work on cobordism, Thom proved a therefore very surprising fact: the cobordism class of an unoriented closed smooth manifold M is completely determined by the so-called Pontryagin numbers, which are computed through Pontryagin classes of the tangent space of M. This was the beginning of a deep revolution in algebraic topology, which contribute to led to generalized cohomology theories, aka *spectra* such as cobordism, complex (real, Morava,...) K-theory, elliptic cohomology, and the beautiful picture painted by chromatic homotopy theory.

In this talk, we will consider the theory of characteristic classes as it was developed in motivic homotopy theory, after Voevodsky, Morel, Panin, Levine, and many more! The authors interest on the subject arose during his PhD under the supervision of Fabien Morel, during the years 1999-2002. This interest has grown during all my carrier (as can be seen in one's bibliography). I would like to seize this opportunity to thank Fabien again to having shared his visions on, and led me to, this wonderful world of motivic homotopy.

1.1. Stable motivic homotopy

1.1.a. **Motivic spectra.**

1.1.1. *Stable homotopy theory.* Recall from the preceding talks that the $\mathbb{A}^1$-homotopy category $\mathrm{H}^{\mathbb{A}^1}(S)$ over a scheme S is obtained by considering the ∞-topos $\mathrm{Sh}^\infty(\mathscr{S}m_S)$ of Nisnevich sheaves over the smooth site $\mathscr{S}m_S$ and by localizing it further with respect to $\mathbb{A}^1$-homotopy: that is we invert for any smooth S-scheme X the maps $\mathbb{A}^1_X \to X$ in $\mathrm{Sh}^\infty(\mathscr{S}m_S)$ via the Yoneda embedding.

On the associated ∞-category $\mathrm{H}^{\mathbb{A}^1}_\bullet(S)$ of pointed objects in $\mathrm{H}^{\mathbb{A}^1}(S)$, we even get a symmetric monoidal ∞-category where the tensor product is the so-called smash product.

We have seen in the talk of F. Morel that there are several models of spheres in motivic homotopy theory: the simplicial sphere S^1, the multiplicative group $(\mathbb{G}_m, 1)$, and the projective line $(\mathbb{P}^1, \infty)$. All objects are considered in $\mathrm{H}^{\mathbb{A}^1}_\bullet(S)$ without indicating the bases scheme S (which plays no specific role here) in the notation. And they are related by the relation:

$$\mathbb{P}^1 \simeq S^1 \wedge \mathbb{G}_m \tag{1.1}$$

[7]This can already be seen in the classification of Seifert fibrations, mentioned above.

As in classical topology, we obtain the stable motivic homotopy category by $\otimes$-inverting the third model of sphere, $\mathbb{P}^1$. For completeness, we will now state the main theorem that will give us our fundamental category (see also the talk of Kirsten Wickelgren).

Theorem 1.1.2 (Robalo). *Let S be any scheme. There exists a universal presentable monoidal ∞-category $\mathrm{SH}(S)$ equipped with a monoidal ∞-functor:*

$$\Sigma^\infty : \mathrm{H}_\bullet^{\mathbb{A}^1}(S) \to \mathrm{SH}(S)$$

which admits a right adjoint Ω^∞ and such that $\Sigma^\infty \mathbb{P}^1$ is $\otimes$-invertible.[8]

Actually, the proposition could be stated more abstractly for an arbitrary presentable monoidal ∞-category and an arbitrary object $\mathbb{S}$. Under, this form the proof is due to Robalo: [Rob15].

1.1.3. It follows from the construction, and the isomorphism (1.1) that, all the possible spheres $\mathbb{S} = \mathbb{P}^1_S, \mathbb{G}_{m,S}, S^1$, becomes $\otimes$-invertible after applying Σ^∞.

The resulting ∞-category acquires a very important property: it is stable in the sense of [Lur17, Def. 1.1.1.9]. This implies that the associated homotopy category admits a triangulated structure (see [Lur17, 1.1.2.13]). Note that the suspension functor for this triangulated structure is given by the formula:

$$\mathbb{E}[1] = \mathbb{E} \otimes \Sigma^\infty S^1.$$

Definition 1.1.4. The monoidal ∞-category $\mathrm{SH}(S)$ is called the *stable motivic homotopy category* over S. Its objects are called *motivic spectra* over S.

The unit object with respect to the monoidal structure is denoted by $\mathbb{1}_S$. One defines the *Tate twist* as $\mathbb{1}_S(1) = \Sigma^\infty \mathbb{P}^1[-2] = \Sigma^\infty \mathbb{G}_{m,S}[-1]$. By construction, this is a $\otimes$-invertible objects in $\mathrm{SH}(S)$ so that one also denotes by $?(n)$ the n-th tensor product with respect to this object.

If this does not cause confusion, we will denote by

$$[\mathbb{E}, \mathbb{F}]_S = \mathrm{Hom}_{\mathrm{SH}(S)}(\mathbb{E}, \mathbb{F}) = \pi_0 \mathrm{Map}_{\mathrm{SH}(S)}(\mathbb{E}, \mathbb{F})$$

the abelian group of morphisms in the homotopy category associated to the ∞-category $\mathrm{SH}(S)$. We usually even drop the index S in the notation.

It might be useful to have in mind the classical model for motivic spectra[9] given here without taking care about the monoidal structure.

[8]An object X in a monoidal ∞-category is $\otimes$-invertible if the ∞-functor $X \otimes ?$ is an equivalence of categories.

[9]That is the objects of an underlying model category whose associated ∞-category is equivalent to the above one.

A model for a motivic spectrum $\mathbb{E}$ is the data of a sequence $(\mathbb{E}_n)_{n\geq 0}$ where $\mathbb{E}_n$ is a pointed simplicial Nisnevich sheaf together with suspension maps:

$$\mathbb{P}^1 \wedge \mathbb{E}_n \to \mathbb{E}_{n+1}.$$

1.1.b. **Representable cohomology theories.** For us, the main function of the stable homotopy category is that its objects, the $\mathbb{P}^1$-spectra, represent cohomology theory. The originality of the theory is that these cohomology theories are *bigraded*.

Definition 1.1.5. *Cohomology theories.* Let $\mathbb{E}$ be a motivic spectrum over S. For any smooth S-scheme X and any pair of integers $(n,i) \in \mathbb{Z}^2$, one defines the $\mathbb{E}$-cohomology of X in degree n and twists i as:

$$\mathbb{E}^{n,i}(X) = [\Sigma^\infty X_+, \mathbb{E}(i)[n]].$$

These cohomologies have the distinctive features of being contravariant, additive, $\mathbb{A}^1$-homotopy invariant and $\mathbb{P}^1$-stable. Moreover, one gets long exact sequences of Mayer-Vietoris type but with respect to Nisnevich distinguished squares.

Example 1.1.6. There are many examples of cohomology theories which are representable in the stable motivic homotopy category, over a given base field $S = \mathrm{Spec}(k)$.

(1) All the classical Weil cohomologies admits canonical extensions over smooth k-schemes which are representable.[10]
   - $\mathrm{char}(k) = 0$: algebraic de Rham cohomology;
   - $\mathrm{char}(k) = p > 0$: rigid cohomology (Berthelot)
   - given an embedding $\sigma : k \subset \mathbb{C}$, the rational singular cohomology of the σ-complex points of a smooth k-scheme X; this is called simply the rational Betti cohomology.
   - give a prime ℓ invertible in k, the $\mathbb{Q}_\ell$-adic étale cohomology of $X \otimes_k \bar{k}$; this is called the geometric $\mathbb{Q}_\ell$-adic cohomology.

   We will denote by $\mathbf{H}_\epsilon$ the spectrum representing one of these Weil cohomologies: $\epsilon =$dR, rig, B, ℓ respectively. In all these cases, twists does not change the cohomology up to an isomorphism (see *loc. cit.* Introduction before theorem 1).

(2) Note that Betti cohomology can be taken with integral coefficients. It is still representable in $\mathrm{SH}(k)$, and twists do not change the cohomology (up to an isomorphism as above). We will denote by $\mathbf{H}_\sigma R$ the corresponding spectrum over k with coefficients in a ring R.

[10]This has been axiomatized in the notion of *mixed Weil cohomology theory* in [CD12].

(3) Given now a real embedding $\sigma : k \subset \mathbb{R}$. One can consider the integral singular cohomology of the real points:

$$H^n(X^\sigma(\mathbb{R}), \mathbb{Z}).$$

This is representable by a ring spectrum that will be denoted by $\mathbf{H}_\sigma \mathbb{Z}$. In that case, twists just shift cohomology degrees, again up to isomorphisms:

$$(\mathbf{H}_\sigma \mathbb{Z})^{n,i}(X) = H^{n-i}(X^\sigma(\mathbb{R}), \mathbb{Z}).$$

Example 1.1.7. The following examples are absolute cohomology in the sense of Beilinson. In the motivic homotopy categorical sense, it means that the base scheme does not really matter when one computes the associated representable cohomology.[11]

(1) Motivic cohomology with coefficients in a ring R, $\mathbf{H}_{\mathrm{M},S}R$ can be defined for any scheme S.[12] The distinctive feature of motivic cohomology is:

$$\mathbf{H}_{\mathrm{M}}^{2n,n}(X, \mathbb{Z}) = \mathrm{CH}^n(X) \qquad X/S \text{ smooth, } S = \text{field, Dedekind ring}$$
$$\mathbf{H}_{\mathrm{M}}^{n,n}(k, \mathbb{Z}) = \mathrm{K}_n^{\mathrm{M}}(k) \qquad k \text{ any field.}$$

where $\mathrm{CH}^n(X)$ denotes the Chow group of X: classes of codimension n algebraic cycles up for the rational equivalence, and $\mathrm{K}_n^{\mathrm{M}}(X)$ is the n-th Milnor K-group: the tensor algebra over the abelian group $k^\times$ modulo the Steinberg relation.

(2) Quillen algebraic K-theory over a regular scheme S. This is represented by a spectrum that we will denote by $\mathbf{KGL}_S$. This spectrum is periodic, in the sense that there exists a canonical isomorphism, the "Bott isomorphism":

$$\beta : \mathbf{KGL}_S(1)[2] \to \mathbf{KGL}_S.$$

Taken into account this isomorphism one gets the following distinctive property for any smooth S-scheme X:

$$\mathbf{KGL}^{n,i}(X) = K_{2i-n}(X)$$

where the right hand-side is Quillen algebraic K-theory: the $(2i-n)$-th homotopy group of the nerve of the Q-category

[11] Concretely, this means that these cohomologies are representable by an *absolute motivic spectrum*: a collection of motivic spectra $\mathbb{E}_S$ over any scheme S, equipped for any morphism $f : T \to S$, with an isomorphism $f^*(\mathbb{E}_S) \simeq \mathbb{E}_T$ satisfying the usual cocycle condition. This is also a cartesian section of the fibred category SH over the category of schemes.

[12] The first definition of such a spectrum is due to Voevodsky. At the time being, one uses a definition based on higher Chow groups and due to Spitzweck. Both definitions coincide if S is smooth over a field.

associated with the exact category of vector bundles over X (pointed by the 0-object).

If one wants a true absolute spectrum, one will replace Quillen K-theory by Weibel homotopy invariant K-theory. We will still denote by $\mathbf{KGL}_S$ the resulting spectrum, for an arbitrary base scheme S.

(3) Algebraic cobordism over any base scheme S, denoted by $\mathbf{MGL}_S$. We will recall later the definition of this spectrum, and state a universal property.

1.1.8. *Extended cohomology.* Representable cohomology theory automatically acquire more structures.For once, a motivic spectrum $\mathbb{E}$ defines a family of contravariant functors:

$$\tilde{\mathbb{E}}^{n,i} : \big(\mathrm{H}_\bullet^{\mathbb{A}^1}(S)\big)^{op} \to \mathscr{A}b, \mathcal{X} \mapsto [\Sigma^\infty \mathcal{X}, \mathbb{E}(i)[n]].$$

Note an important property of this functor: it turns cofiber sequences in $\mathrm{H}_\bullet^{\mathbb{A}^1}(S)$ into long exact sequences of abelian groups.[13]

An interesting remark is that the $\mathbb{E}$-cohomology is therefore invariant under weak motivic equivalences.[14]

Secondly, one immediately gets a definition of cohomology with support. A closed S-pair (X, Z) will be a pair of schemes such that X is a smooth S-scheme, and $Z \subset X$ a closed subscheme. By taking homotopy cofibers, in the pointed motivic homotopy category, one can define the object $X/X - Z$ which fits into a cofiber sequence:

$$(X - Z)_+ \xrightarrow{j_*} X_+ \to X/X - Z$$

One defines the $\mathbb{E}$-cohomology of X with support in Z in degree n and twist i as:

$$\mathbb{E}_Z^{n,i}(X) := \tilde{\mathbb{E}}^{n,i}(X/X - Z).$$

Therefore, it fits into a long exact sequence:

$$\ldots \mathbb{E}_Z^{n,i}(X) \to \mathbb{E}^{n,i}(X) \xrightarrow{j^*} \mathbb{E}^{n,i}(X - Z) \xrightarrow{\partial_{X,Z}} \mathbb{E}_Z^{n+1,i}(X) \ldots$$

This cohomology with support enjoys good properties:

(1) Contravariance: for any morphism $f : Y \to X$ of smooth S-schemes, there exists a pullback functor:

$$f^* : \mathbb{E}_Z^{n,i}(X) \to \mathbb{E}_{f^{-1}(Z)}^{n,i}(Y).$$

[13] This comes from the fact that Σ^∞, as a left adjoint, preserves cofiber sequences. As its target is a stable ∞-category, it even sends cofiber sequences to exact sequences.

[14] To get a nice picture on weak motivic equivalences, we refer the survey paper [AOsr21].

(2) Covariance: for any closed immersion $i : T \to Z$ of closed subschemes of X, one gets:

$$i_* : \mathbb{E}^{n,i}_T(X) \to \mathbb{E}^{n,i}_Z(X).$$

Remark 1.1.9. (1) All the previous examples admit a natural notion of cohomology with support, which agree with the above definition.

(2) Morel-Voevodsky purity theorem implies the following property of cohomology with support, in the case where $Z \subset X$ is a smooth subscheme of codimension c:

$$\mathbb{E}^{n,i}_Z(X) \simeq \mathbb{E}^{n-2c,i-c}(Z).$$

1.1.c. **Ring spectra and cup-products.** The next definition is the last piece of structure one needs on cohomology to get characteristic classes.

Definition 1.1.10. A *(commutative) ring spectrum* $\mathbb{E}$ over the base scheme S is a (commutative) monoid object in the homotopy category associated with $\mathrm{SH}(S)$.

In particular, the structure of a ring spectrum on $\mathbb{E}$ is given by a unit $1_{\mathbb{E}} : \mathbb{1}_S \to \mathbb{E}$ and a product $\mu : \mathbb{E} \otimes_S \mathbb{E} \to \mathbb{E}$, which satisfies the usual axioms. If one wants to be precise, we will say that $(\mathbb{E}, \mu, 1_{\mathbb{E}})$ is a motivic ring spectrum.

One deduces a product on $\mathbb{E}$-cohomology, which is often called the cup-product[15]: given cohomology classes:

$$a : \Sigma^\infty X_+ \to \mathbb{E}(i)[n], b : \Sigma^\infty X_+ \to \mathbb{E}(j)[m]$$

one defines $a \cup_\mu b$ as the composite map:

$$\Sigma^\infty X_+ \xrightarrow{\delta_*} \Sigma^\infty(X \times_S X)_+ = \Sigma^\infty X_+ \otimes_S \Sigma^\infty X_+ \xrightarrow{a \otimes b} \mathbb{E} \otimes \mathbb{E}(i+j)[n+m] \xrightarrow{\mu} \mathbb{E}(i+j)[n+m].$$

We will usually denote this product simply as ab.

It follows that for any smooth S-scheme X, $\mathbb{E}^{**}(X)$ is a bi-graded algebra over the bigraded ring $\mathbb{E}^{**}(S)$, usually called the *coefficient ring* of $\mathbb{E}$ and simply denoted by $\mathbb{E}^{**}$.

Remark 1.1.11. One should be careful that the above bigraded algebra is not simply graded commutative with respect to the first index. To state the required formula one neesd the special element $\epsilon \in [\mathbb{1}_S, \mathbb{1}_S]$, which acts as a scalar on any representable cohomology theory, defined by the switch map inverse map $x \mapsto x^{-1}$ on $\mathbb{G}_m$, and using that $\Sigma^\infty(\mathbb{G}_m, 1) = \mathbb{1}_S(1)[1]$.

[15] This terminology, due to Whitney for the product on singular cohomology, has firmly remained in algebraic topology, due to the tremendous importance of its introduction in the thirties.

Then the ϵ-graded commutativity formula, for a, b as above, reads as follows:

$$ab = (-1)^{n+m-i-j}.\epsilon^{i+j}.ba \tag{1.2}$$

The proof is formal once one notices that ϵ can also be defined, up to $\mathbb{A}^1$-homotopy, by the map switching the factors on $\mathbb{G}_m \times \mathbb{G}_m$ (see [Mor04, Lemma 6.1.1]).

Example 1.1.12. All the examples of cohomology theories of Example 1.1.6 and Example 1.1.7 are in fact representable by motivic ring spectra, and the associated cup-product corresponds to their usual product.

Remark 1.1.13. The theory developed below only requires the above definition. However, all the examples considered admits a *highly structured product*, *i.e.* it is the object in the homotopy category associated with a (commutative) algebra object of the monoidal ∞-category $\mathrm{SH}(S)$. Beware that in general, it is fundamental in classical (and motivic) stable homotopy theory to give a clear distinction between those two kinds of structure.

1.1.d. **Representability of the Picard group.**

1.1.14. In algebraic topology, given an abstract group G, one can define its classifying space BG as an explicit simplicial set: the nerve of the groupoid associated with G, made of a single object $*$, a morphism for any element of g, the composition being given by the group law.

It is more common to consider topological groups (eg. Lie groups) G, and then one can still define a classifying space BG (as an explicit CW-complex) with the distinctive feature that for any topological space X, the homotopy classes of (unpointed) maps $[X, BG]$ are in bijhection with the principal homogeneous G-spaces.

In motivic homotopy theory, Morel and Voevodsky have provided an analog of the second construction but using the first one and the framework of simplicial sheaves. For an algebraic group G over a scheme S, and a smooth S-scheme X, we denote by $H^1_{\mathrm{Nis}}(X, G)$ the set of G-torsors on X for the Nisnevich topology. Let us state a particular case of Morel-Voevodsky's construction relevant in our case.

Proposition 1.1.15. *Let S be a scheme, and G be an algebraic group over S.*

Then there exists an object BG in $\mathrm{H}_\bullet^{\mathbb{A}^1}(S)$ and for any smooth S-scheme X, a canonical functorial application of (pointed) sets:

$$H^1_{\mathrm{Nis}}(X, G) \to [X, BG]^{un}_S. \tag{1.3}$$

Moreover, if the left hand-side is $\mathbb{A}^1$-invariant over all smooth S-schemes, this map is an isomorphism.

Note that it is very rare that the second condition holds. The only example we have in mind is that of $G = \mathbb{G}_m$ when S is regular. To sum-up in the case of $\mathbb{G}_m$, we get a canonical map:

$$\mathrm{Pic}(X) \to [X, B\mathbb{G}_{m,S}]_S^{un}$$

which is bijective whenever S is regular. Note also that the theory of Morel and Voevodsky shows that $B\mathbb{G}_{m,S}$ admits the geometric model that one expect: it is the infinite projective space $\mathbb{P}_S^\infty$, that is the infinite Grassmanian of lines in an affine space:

$$B\mathbb{G}_{m,S} = \varinjlim_{n\geq 0} \mathbb{P}_S^n$$

where the colimit can be taken in the category of simplicial sheaves on $\mathscr{S}m_S$ (to get an explicit model). Note that $\mathbb{P}_S^\infty$ will be seen as a pointed sheaves via the point at ∞ of all the $\mathbb{P}_S^n$. We will recall from this discussion the canonical map:

$$\mathrm{Pic}(X) \to [X, \mathbb{P}_S^\infty]_S^{un}. \tag{1.4}$$

Remark 1.1.16. (1) Assume $S = \mathrm{Spec}(k)$ is the spectrum of a field. If one restricts our attention to smooth affine k-schemes X, then the map (1.3) is an isomorphism for an isotropic reductive k-group schemes: e.g. $G = \mathrm{GL}_n, \mathrm{SL}_n, \mathrm{Sp}_{2n}$. This is a theorem which was first obtained by Morel in certain cases, and in general by Asok, Hoyois and Wendt (see [AHW20] for the extra condition needed for G).

(2) Morel and Voevodsky also give geometric models for over classifying spaces: as an example, for any $n \geq 0$, BGL_n is equivalent to the infinite grassmanian of sub-n-vector bundles:

$$\mathrm{BGL}_n = \varinjlim_{r\geq 0} \mathrm{Gr}(n, n+r).$$

1.2. Oriented ring spectra

1.2.a. **Definition and examples.** Given the notation of the previous section, we have all the ingredients to formulate the notion of orientation, which has been introduced in motivic homotopy theory, by analogy with topology, at the time of the first proof of the Milnor conjecture by Voevodsky.

Definition 1.2.1. Let $(\mathbb{E}, \mu, 1_\mathbb{E})$ be a ring spectrum over S . Let $i : \mathbb{P}_S^1 \to \mathbb{P}_S^\infty$ be the canonical inclusion of pointed Nisnevich sheaves, both being pointed by the point at ∞.

An *orientation* of $\mathbb{E}$ is the data of a class $c \in \tilde{\mathbb{E}}^{2,1}(\mathbb{P}^\infty_S)$ suth that $i^*(c) = 1_\mathbb{E}$ via the identification: $\tilde{\mathbb{E}}^{2,1}(\mathbb{P}^1_S) = \mathbb{E}^{0,0}(S)$.
We will say that the pair $(\mathbb{E}, c)$ is an oriented (ring) spectrum.

Note that an orientation can be seen as a map:

$$c : \Sigma^\infty \mathbb{P}^\infty_S \to \mathbb{E}(1)[2].$$

Example 1.2.2. (1) Let X be a smooth $\mathbb{Z}$-scheme. We have seen that there is an isomorphism: $\mathbf{H}^{2,1}_{\mathrm{M}}(X) \simeq \mathrm{CH}^1(X) \simeq \mathrm{Pic}(X)$. This extends to ind-smooth $\mathbb{Z}$-schemes. But $\mathrm{Pic}(\mathbb{P}^\infty_S) \simeq \mathbb{Z}.c$, the free abelian group generated by c, the class of the tautological invertible bundle $\lambda = \mathcal{O}_{\mathbb{P}^\infty_S}(-1)$.

Moreover, the restriction of c to $\mathbb{P}^1_S$ is the cycle class of the point at ∞. It is the unit of the ring structure on $\mathrm{CH}^*(\mathbb{P}^1_S) \simeq \mathbf{H}^{2*,*}_{\mathrm{M}}(\mathbb{P}^1_S) \simeq \mathbb{Z}[c]$.[16] Therefore, the class c corresponds to an orientation of $\mathbf{H}_{\mathrm{M}}\mathbb{Z}$ over the base scheme $\mathbb{Z}$.

Given now any scheme X, we can look at the canonical map $f : S \to \mathbb{Z}$. Then f^*, being compatible with products on motivic cohomology, $f^*(c)$ is an orientation of $f^*\mathbf{H}_{\mathrm{M}}\mathbb{Z} = \mathbf{H}_{\mathrm{M}}\mathbb{Z}_S$.

(2) Let $\mathbf{H}_\epsilon$ be the ring spectrum representing one of the mixed Weil cohomology theories, over smooth k-schemes with coefficient in the appropriate field K of characteristic 0 as in Example 1.1.6. The corresponding cohomology admits a cycle class map:

$$\mathrm{CH}^i(X) \simeq \mathbf{H}^{2i,i}_{\mathrm{M}}(X) \to \mathbf{H}^{2i,i}_\epsilon(X) \overset{(*)}{\simeq} \mathbf{H}^{2i,0}_\epsilon(X) =: \mathbf{H}^{2i}_\epsilon(X)$$

which is compatible with products (mapping intersection products to "cup-products").[17] Therefore, the image of $c \in \mathrm{CH}^1(\mathbb{P}^\infty_k)$ in $\mathbf{H}^{2,1}_\epsilon(\mathbb{P}^\infty_S)$ induces a canonical orientation of the mixed Weil spectrum $\mathbf{H}_\epsilon$.

(3) The same strategy works for the singular cohomology of complex points of smooth algebraic k-schemes, over $k \subset \mathbb{C}$.

(4) On the contrary, the ring spectrum representing singular cohomology of the real points of smooth algebraic k-scheme, $k \subset \mathbb{R}$, is not orientable. Indeed:

$$\mathbf{H}^{2,1}_\sigma(\mathbb{P}^1_k) = H^1(\mathbb{RP}^1, \mathbb{Z}) = \mathbb{Z}$$
$$\mathbf{H}^{2,1}_\sigma(\mathbb{P}^\infty_k) = H^1(\mathbb{RP}^\infty, \mathbb{Z}) = \mathbb{Z}/2.$$

[16] This follows for example from the definition via pullback along the diagonal.

[17] We put the last isomorphism to recall that mixed Weil cohomologies are $(0,1)$-periodic. This isomorphism is non-canonical and depends on the choice of a generator of the 1-dimensional K-vector space $\mathbf{H}^{1,1}_\epsilon(\mathbb{G}_{m,k})$. See [CD12] as indicated in Example 1.1.6.

(5) The spectrum **KGL** representing algebraic K-theory is oriented by the following class:

$$c^{\mathbf{KGL}}(L) = \beta^{-1}(1 - [\lambda^\vee]) \in \mathbf{KGL}^{2,1}(\mathbb{P}^\infty_S)$$

where we denote by $[L] \in \mathbf{KGL}^{0,0}(X) \simeq K_0(X)$ the class of a line bundle L/X in the Grothendieck group of vector bundles over X, and $\beta \in \mathbf{KGL}^{-2,-1}(S)$ is the Bott element (over S).

(6) The algebraic cobordism spectrum $\mathbf{MGL}_S$ admits, by construction, a canonical orientation. That will be cleared out in the next course.

1.2.b. **Chern classes.**

1.2.3. *First Chern class.* Let $(\mathbb{E}, c)$ be an oriented ring spectrum over S. Taken into account the canonical map (1.4), we obtain for any smooth S-scheme X a canonical map:

$$\begin{aligned} \mathrm{Pic}(X) \to [X_+, \mathbb{P}^\infty_S]_{\mathrm{H}^{\mathbb{A}^1}_\bullet(S)} &\xrightarrow{\Sigma^\infty} [\Sigma^\infty X_+, \Sigma^\infty \mathbb{P}^\infty_S]_{\mathrm{SH}(S)} \\ &\xrightarrow{c_*} [\Sigma^\infty X_+, \mathbb{E}(1)[2]]_{\mathrm{SH}(S)} = \mathbb{E}^{2,1}(X). \end{aligned}$$

This is called the first Chern class associated with the orientation c, denoted simply by c_1. It is clearly contravariantly functorial in the scheme X. However we must observe at this point that c_1 is simply an application, and not necessarily a morphism of groups. In fact, all the maps in the above compositum are morphisms of groups *except* the suspension map Σ^∞. This fact is extremely meaningful in the theory of oriented ring spectra (see the next course).

The key fact of the theory is the following *projective bundle formula*:

Theorem 1.2.4. *Consider the above notation. Let $V \to X$ be a rank n vector bundle over a smooth S-scheme X, and let $P = \mathbb{P}(V)$ be the associated projective bundle. We let $p : P \to X$ be the canonical projection, and let λ_P be the canonical line bundle on P (coming from the fact $\mathbb{P}(V)$ classifies sub-line bundles of V).*[18] *Then the following map:*

$$\begin{aligned} \oplus_{i=0}^{d-1} \mathbb{E}^{**}(X) &\to \mathbb{E}^{**}(P) \\ \lambda_i &\mapsto \sum_i p^*(\lambda_i).c_1(\lambda_P)^i \end{aligned}$$

is an isomorphism of $\mathbb{E}^{**}(X)$-modules.

[18] This line bundle is often denoted by $\mathcal{O}_P(-1)$, for example by Fulton in [Ful98].

One can reformulate the above theorem by saying that $\mathbb{E}^{**}(P)$ is a bigraded $\mathbb{E}^{**}(X)$-algebra (through the pullback map p^*) which is free of rank n, generated by $c_1(\lambda_P)^i$ for $0 \leq i \leq r-1$.

Remark 1.2.5. *Milnor sequence.* In general, for a ring spectrum $(\mathbb{E}, \mu, 1_{\mathbb{E}})$ over S, one always has the so-called Milnor exact sequence:

$$0 \to \varprojlim_{n\geq 0}{}^1 \mathbb{E}^{2,1}(\mathbb{P}^n_S) \to \mathbb{E}^{2,1}(\mathbb{P}^\infty_S) \to \varprojlim_{n\geq 0} \mathbb{E}^{2,1}(\mathbb{P}^n_S) \to 0.$$

It follows from the above that, whenever $\mathbb{E}$ is oriented, the left hand side vanishes as the involved inductive system satisfies the Mittag-Leffler condition. In particular, to give an orientaition on $\mathbb{E}$, it is sufficient to give classes $c_n \in \mathbb{E}^{2,1}(\mathbb{P}^n_S)$ for all $n > 0$ such that $c_1 = 1_{\mathbb{E}}$ and $\iota_n^*(c_{n+1}) = c_n$.

As a corollary, one gets our first family of characteristic classes, the Chern classes of algebraic vector bundles, following a method of Grothendieck.

Definition 1.2.6. Let $(\mathbb{E}, c)$ be an oriented (motivic) ring spectrum over S. Let X be a smooth S-scheme and V/X be a vector bundle or rank n. Then there exists a unique family $(c_i(V))_{0\leq i\leq n}$ such that the following relation holds in $\mathbb{E}^{2,1}(\mathbb{P}(V))$:

$$\sum_{i=0}^{n} p^*\big(c_i(V)\big).\big(-c_1(\lambda_P)\big)^{n-i}$$

Note in particular that $c_i(V) \in \mathbb{E}^{2i,i}(X)$. If $i > n$, we put $c_i(V) = 0$.

1.2.7. According to the above definition, we get the following properties of Chern classes:

(1) *Invariance under isomorphism.* For any isomorphism $V \simeq V'$ of vector bundles over X, $c_i(V) = c_i(V')$.
(2) *Compatibility with pullbacks.* For any vector bundle V/X, and any morphism $f : Y \to X$ of smooth S-schemes, $f^*c_i(V) = c_i(f^{-1}V)$.
(3) *Triviality.* For a trivializable vector bundle V, $c_i(V) = 0$ if $i > 0$.
(4) *Nilpotence.* Here it is important that S is noetherian. For any vector bundle V/X, and any $i \geq 0$, the Chern class $c_i(V)$ is nilpotent.

The third relation follows from the fact $c_1(\mathcal{O}_{\mathbb{P}^n}(-1))^{n+1} = 0$ (see the proof of the projective bundle theorem). The last relation is left as an exercice to the reader.

To go further, one needs the so-called *splitting principle*. It is based on the following "splitting construction".

Proposition 1.2.8. *Let X be a smooth S-scheme, and V a vector bundle over X. Then there exists a smooth projective map $p : X' \to X$ such that $p^{-1}(V)$ splits as a direct sum of line bundles and such that for any oriented ring spectrum $\mathbb{E}$ over S, the pullback map $p^* : E^{**}(X) \to \mathbb{E}^{**}(X')$ is injective.*

Remark 1.2.9. A canonical construction for X' is to take the flag bundle associated with V, which is the moduli space which parametrize the complete flag of sub-vector bundles of V. The fact the projection map induces an injective pullback on an oriented cohomology theory can be seen as a *motivic Leray-Hirsch theorem*. The latter can be obtained directly from the homotopy Leray spectral sequence of [ADN20] associated with p and with coefficients in $\mathbb{E}$.

1.2.10. *Splitting principle.* As a corollary of the previous proposition, one obtains the so-called splitting principle for Chern classes associated with any oriented ring spectrum $(\mathbb{E}, c)$ as above. Let V/X be a rank n vector bundle over a smooth S-scheme X.

First, we define the total Chern class as the polynomial in t, with coefficients in the (bigraded) ring $\mathbb{E}^{**}(X)$:[19]

$$c_t(V) = \sum_{i \geq 0} c_i(V).t^i.$$

Then the splitting principle tells us that, to compute with the Chern classes of V, one can assume that V is split using the preceding splitting construction. this amounts to say that the total Chern class splits: it admits Chern roots α_i such that:

$$c_t(V) = \prod_{i=1}^{n} (1 + \alpha_i.t)$$

Then any symmetric polynomial in the Chern roots α_i admits an expression in terms of the Chern classes of V.

As an example, one can get the formula:

Proposition 1.2.11 (Whitney sum formula)**.** *For any exact sequence of vector bundles over a smooth S-scheme X:*

$$0 \to V' \to V \to V'' \to 0$$

[19] This convention for total Chern class follows Fulton [Ful98]. Other conventions, such as for example in [MS74] simply considers the sum $c(V) = \sum_i c_i(V)$ in the "total" cohomology $\bigoplus_i \mathbb{E}^{2i,i}(X)$.

one has: $c_t(V) = c_t(V').c_t(V'')$.

Example 1.2.12. Consider the above notation. Given V/X a vector bundle of rank n, one usually defines the *Euler class* of V as:

$$e(V) = c_n(V).$$

Assume X is smooth affine of dimension n over $S = \mathrm{Spec}(k)$, the spectrum of an algebraically closed field (or a field in which (-1) is a sum of squares). Then we have seen in the talk of Aravind Asok that the vanishing of the Euler class in motivic cohomology, is equivalent to the fact V splits-off a trivial summand. (One direction obviously follows from the above Whitney sum formula!) We refer the interested reader to the milestone ICM talk of Aravind and Jean: [AF23].

However, to remove the assumption on k, one needs a finer version of the Euler class, with values in the Chow-Witt group.

1.2.c. **The algebraic Hopf map.**

1.2.13. The endomorphism ring of the sphere spectrum, $\mathrm{End}(\mathbb{1}_S)$ acts on any motivic spectrum $\mathbb{E}$. Similarly, any map $\varphi : \mathbb{1}_S \to \mathbb{1}_S(i)[n]$ induces a morphism $\varphi \otimes \mathbb{E} : \mathbb{E} \to \mathbb{E}(i)[n]$. This can be seen as an action of the graded ring $\Pi^{n,i}(S)$ — the stable motivic cohomotopy of S — on $\mathbb{E}$.

According to the fundamental theorem of Morel, when S is the spectrum of a field k, one gets $\Pi^{n,n}(k) \simeq \mathrm{K}_n^{\mathrm{MW}}(k)$, the Milnor-Witt ring of k. Other any base S, one still gets important endomorphisms:

(1) *Algebraic Hopf map.* $\eta : \mathbb{1}_S(1)[1] \to \mathbb{1}_S$, which is induced by the canonical map $\mathbb{A}^2_S - \{0\} \to \mathbb{P}^1_S, (x, y) \mapsto [x : y]$ (in coordinates).

(2) *Classes of units.* for any $u \in \mathcal{O}(S)^\times$, one deduces $[u] : \mathbb{1}_S \to \mathbb{1}_S(1)[1]$ from the map $u : S \to \mathbb{G}_{m,S}$ corresponding to u. One then puts:
$$< u >= 1 + \eta.[u],$$
which is an element in degree $(0, 0)$ of the bigraded ring $\Pi^{**}(S)$.

Note that one can check that $\epsilon = - < -1 > \in \Pi^{0,0}(S)$, where ϵ was defined in Remark 1.1.11.[20]

As a consequence of the projective bundle theorem and using the above mentioned remark, one deduces:

Proposition 1.2.14. *Let* $\mathbb{E}$ *be an orientable ring spectrum. Then the algebraic Hopf map* η *acts trivially on* $\mathbb{E}$*:* $\eta \otimes \mathbb{E} = 0$.

[20] In fact, η and ϵ are defined by pullbacks from elements of $\Pi^{**}(\mathbb{Z})$. It is likely that $\Pi^{0,0}(\mathbb{Z}) = \mathbb{Z}[\epsilon]/(\epsilon^2 = 1)$. This would be a direct consequence of the absolute purity property for (reduced) closed subschemes of $\mathrm{Spec}(\mathbb{Z})$.

As a consequence, for every units $u \in \mathcal{O}(S)^\times$, $< u >$ *acts by the identity. In particular,* ϵ *acts by* (-1): $\epsilon \otimes \mathbb{E} = -\operatorname{Id}_{\mathbb{E}}$. *As a consequence, relation* (1.2) *becomes*

$$ab = (-1)^{nm}.ba.$$

Proof. The first assertion follows from the cofiber sequence in the pointed motivic homotopy category:

$$\mathbb{A}^2 - \{0\} \xrightarrow{\eta} \mathbb{P}^1_S \xrightarrow{\iota_1} \mathbb{P}^2_S$$

for which we refer to [Mor04]. Indeed, if $\mathbb{E}$ is oriented, then $\mathbb{E} \otimes \iota_1$ is a split monomorphism. The rest of the assertions follow easily. □

In general, the action of the Hopf map is not sufficient to detect orientability of a ring spectrum. However, we have the notable theorems.

Theorem 1.2.15. *Let* k *be a perfect field, and* $\mathbb{E} \in \mathrm{SH}(k)$ *be a homotopy module with a ring structure. Then the following conditions are equivalent:*

(i) $\mathbb{E}$ *is orientable.*
(ii) $\eta \otimes \mathbb{E} = 0$.
(iii) $\mathbb{E}$ *admits transfers in the sense of Voevodsky (*i.e. *action of finite correspondences).*

This theorem uses the equivalence between homotopy modules with transfers and Rost cycle modules: see [Dég13]. We can now obtain a more direct proof by using the equivalence of homotopy modules with Milnor-Witt cycle modules: see [Fel21].

Theorem 1.2.16 (Morel, Cisinski-D.)**.** *Let* $\mathbb{E}$ *be a rational motivic ring spectrum over a scheme* S. *Then the following conditions are equivalent:*

(i) $\mathbb{E}$ *is orientable.*
(ii) $\eta \otimes \mathbb{E} = 0$.
(iii) $\epsilon \otimes \mathbb{E} = -\operatorname{Id}_{\mathbb{E}}$.

In fact in these case, $\mathbb{E}$ *is a rational motive !*

Sketch of proof.[21] The proof relies on Morel's decomposition of the rational stable homotopy category into:

$$\mathrm{SH}(S)_{\mathbb{Q}} \simeq \mathrm{SH}(S)_{\mathbb{Q}+} \times \mathrm{SH}(S)_{\mathbb{Q}-}$$

characterized by the equivalent properties:

(i) $\mathbb{E} \in \mathrm{SH}(S)_{\mathbb{Q}+}$ (resp. $\mathbb{E} \in \mathrm{SH}(S)_{\mathbb{Q}-}$).
(ii) $\epsilon \otimes \mathbb{E}$ is equal to -1 (resp. $+1$).

[21] This proof is a simplification of the proof given in [CD19, Th. 16.2.13].

(iii) $\eta \otimes \mathbb{E}$ is null (resp. invertible).

Then the main point is to show that the canonical map:

$$\mathbb{1}_S \otimes \mathbb{Q}_+ \to \mathbf{H}_{\mathrm{B},S}$$

is an isomorphism, where the right hand-side is Beilnson motivic cohomology ring spectrum (representing the 0-th graded piece of rational algebraic K-theory, over regular schemes). By a localization arguments and invariance under inseparable field extensions, one reduces to the case of a perfect field k. Then a devissage argument (we use rational coefficients at this point) reduces to the preceding theorem.

Remark 1.2.17. As a complement, let us say that one now knows how to compute both the plus and the minus part of rational motivic stable homotopy category (see [DFJK21]):

$$\mathrm{SH}(S)_{\mathbb{Q}+} \simeq \mathrm{DM}(S, \mathbb{Q})$$

is the category of rational mixed motivic complexes. In particular, rationally, being orientable is the same as being a motivic complex.

For the minus part, one has:

$$\mathrm{SH}(S)_{\mathbb{Q}-} \simeq \mathbf{H}\,\underline{\mathrm{W}}_{S \otimes_{\mathbb{Z}} \mathbb{Q}} - \mathrm{mod}$$

where the right hand-side is the category of modules over the unramified rational Witt sheaf, seen over the caracteristic 0 part $S \otimes_{\mathbb{Z}} \mathbb{Q}$ of S (in particular, it is zero on a scheme of positive charadteristic).

Cours 2. Oriented spectra: Thom classes and formal group laws

2.1. Thom classes

2.1.a. Construction.

2.1.1. Let V be a rank n vector bundle over a smooth S-scheme X, and $(\mathbb{E}, c)$ be an oriented ring spectrum over S. We let $\nu : \mathbb{P}(V) \to \mathbb{P}(V \oplus \mathbb{A}^1)$ be the canonical closed immersion.[22]

Recall from the talk of Kirsten Wickelgren that one defines the Thom space of V in the pointed motivic homotopy category $\mathrm{H}^{\mathbb{A}^1}_{\bullet}(S)$ over S as the following homotopy cofibers:

$$\mathrm{Th}_S(V) = V/V^\times = \mathbb{P}(V \oplus \mathbb{A}^1)/\mathbb{P}(V).$$

[22]The target of ν is known as the projective completion of V: the open complement of ν is isomorphic to V.

In general, it is clear over which base one considers the Thom space $\mathrm{Th}_S(V)$ so that we will denote it simply by $\mathrm{Th}(V)$.[23]

One deduces from the above formula for Thom spaces a long exact sequence

$$\ldots \to \tilde{\mathbb{E}}^{**}(\mathrm{Th}(V)) \xrightarrow{\pi^*} \mathbb{E}^{**}(\mathbb{P}(V \oplus \mathbb{A}^1)) \xrightarrow{\nu^*} \mathbb{E}^{**}(\mathbb{P}(V)) \to \ldots \tag{2.5}$$

where we have used the extension of $\mathbb{E}^{**}$ described in 1.1.8.[24]

It follows from Theorem 1.2.4 that the map ν^* is a split epimorphism of free $\mathbb{E}^{**}(X)$-modules of respective ranks n and $n-1$. Thus $E^{**}(\mathrm{Th}(X))$ is a free $\mathbb{E}^{**}(X)$-module of rank 1, isomorphic to $\ker(\nu^*)$. One deduces from this discussion the following definition.

Definition 2.1.2. Consider the above notation and assumptions.

We define the *Thom class* of V/X as the following element of $\mathbb{E}^{2n,n}(\mathbb{P}(V \oplus \mathbb{A}^1))$:

$$\mathrm{th}(V) = \sum_{i=0}^{n} p^*(c_i(V)).\big(-c_1(\lambda)\big)^{n-i}$$

using the notation of Theorem 1.2.4. We define the *refined Thom class* $\overline{\mathrm{th}}(V)$ of V as the unique element of $\mathbb{E}^{2n,n}(\mathrm{Th}(V))$ such that

$$\pi^*\big(\overline{\mathrm{th}}(V)\big) = \mathrm{th}(V).$$

It follows from the split exact sequence (2.5) of $\mathbb{E}^{**}(X)$-modules that the following map

$$\mathbb{E}^{**}(X) \to \mathbb{E}^{**}(\mathrm{Th}(V)), \lambda \mapsto \lambda.\overline{\mathrm{th}}(V) \tag{2.6}$$

is an isomorphism of bidegre $(2n, n)$, called the *Thom isomorphism* associated with the vector bundle V/X and with coefficients in the oriented ring spectrum $(\mathbb{E}, c)$.

We deduce from the analogous properies of Chern classes that Thom classes are compatible with base change and invariant under isomorphisms of vector bundles.

Example 2.1.3. Recall the universal quotient bundle ξ on $\mathbb{P}(V \oplus \mathbb{A}^1)$ is defined by the exact sequence

$$0 \to \mathcal{O}(-1) \to p^{-1}(V \oplus 1) \to \xi \to 0.$$

[23]Beware however it could also be considered as an object of $\mathrm{H}_\bullet^{\mathbb{A}^1}(X)$, which is reflected in the more precise notation $\mathrm{Th}_X(V)$.

[24]Recall that $\tilde{\mathbb{E}}^{**}(\mathrm{Th}(V))$ can also be described as the $\mathbb{E}$-cohomology of V with support in the 0-section of V/X.

Thus the Whitney sum formula Proposition 1.2.11 gives the following relation between Thom and Chern classes:

$$\mathrm{th}(V) = c_n(\xi) = e(\xi). \tag{2.7}$$

2.1.4. It will be useful to work internally, and relatively. Consider the notation of the previous definition and let $p : X \to S$ be a smooth morphism.

Recall we have the base change functor $p^* : \mathrm{SH}(S) \to \mathrm{SH}(X)$ with left adjoint $p_\sharp$. We put $\mathbb{E}_X = p^*(\mathbb{E})$. Note the orientation of $\mathbb{E}$ determines a canonical orientation of $\mathbb{E}_X$. The object $\mathbb{E}_X$ is a commutative monoid of $\mathrm{Ho}\,\mathrm{SH}(X)$ so we can consider the additive category of $\mathbb{E}_X$-modules. Then the Thom isomorphism associated with a vector bundle V of rank r over X actually corresponds to an isomorphism of $\mathbb{E}_X$-modules:

$$\mathrm{th}(V) : \mathbb{E}_X \otimes \Sigma^\infty \mathrm{Th}(V) \to \mathbb{E}_X(r)[2r].$$

2.1.b. **Universal property of algebraic cobordism.**

2.1.5. *Construction of algebraic cobordism.* Let us now recall the construction of the algebraic cobordism spectrum $\mathbf{MGL}_S$ (which is modeled on the construction of the topological spectra MU, MO, ...).

Though all objects will be considered over S, we drop the index in the notation for simplicity. We consider the tautological rank n vector bundle γ_n on the classifying space BGL_n viewed as a smooth S-scheme via the model given by the infinite grasmannian:

$$\mathrm{BGL}_n = \mathrm{colim}_{m \geq n} \mathrm{Gr}_n(\mathbb{A}^m).$$

As $\gamma_n \oplus \mathbb{A}^r$ as rank $n + r$, one deduces a (homotopy) cartesian square:

$$\begin{array}{ccc} \gamma_n \oplus \mathbb{A}^r & \xrightarrow{(*)} & \gamma_{n+r} \\ \downarrow & & \downarrow \\ \mathrm{BGL}_n & \longrightarrow & \mathrm{BGL}_{n+r}. \end{array}$$

One deduces from $(*)$ a canonical map of Thom spaces in $\mathrm{H}^{\mathbb{A}^1}_\bullet(S)$:

$$\mathrm{Th}(\gamma_n)(r)[2r] = \mathrm{Th}(\gamma_n \oplus \mathbb{A}^r) \to \mathrm{Th}(\gamma_{n+r}).$$

In particular, one deduces a tower in $\mathrm{SH}(S)$:

$$\Sigma^\infty \mathrm{Th}(\gamma_0) \to \Sigma^\infty \mathrm{Th}(\gamma_1)(-1)[-2] \to \\ \ldots \to \Sigma^\infty \mathrm{Th}(\gamma_n)(-n)[-2n] \to \ldots$$

One defines the algebraic cobordism spectrum as the homotopy colimit:

$$\mathbf{MGL}_S = \mathrm{hocolim}_{n \geq 0} \Sigma^\infty \mathrm{Th}_S(\gamma_n)(-n)[-2n]. \tag{2.8}$$

Further, using the canonical map $\gamma_n \times \gamma_m \to \gamma_{n+m}$ one can build ring structure on $\mathbf{MGL}_S$.[25]

Note that the vector bundle γ_1 over $B\mathbb{G}_{m,S}$ can be identified with the canonical line bundle $\mathcal{O}_{\mathbb{P}^\infty}(-1)$ on $\mathbb{P}^\infty$ via the weak $\mathbb{A}^1$-equivalence $B\mathbb{G}_{m,S} \simeq \mathbb{P}^\infty_S$. We need the following lemma.

Lemma 2.1.6. *Consider the above notion. Then there exists a canonical weak $\mathbb{A}^1$-homotopy equivalence of pointed motivic spaces over S:*

$$\mathbb{P}^\infty_S \simeq \mathrm{Th}(\gamma_1).$$

Proof. We consider the closed immersion:

$$\mathbb{P}^{n-1}_S \xrightarrow{\iota_n} \mathbb{P}^n_S.$$

The normal bundle of ι_n is the canonical line bundle $\mathcal{O}_{\mathbb{P}^{n-1}}(-1)$ on $\mathbb{P}^{n-1}$. Moreover, the open complementary $\mathbb{P}^n_S - \mathbb{P}^{n-1}_S$ is isomorphic to the affine line $\mathbb{A}^n_S$, so it is contractible. Using Morel-Voevodsky's purity theorem, one gets:

$$(\mathbb{P}^n_S, 1) \simeq \mathbb{P}^n_S/(\mathbb{P}^n_S - \mathbb{P}^{n-1}_S) \simeq \mathrm{Th}(\mathcal{O}_{\mathbb{P}^{n-1}})(-1)).$$

It is important to note that this isomorphism is functorial with respect to the inclusion ι_n. Therefore, one can take the homotopy limit over n, and this defines the required isomorphism. □

2.1.7. *Canonical orientation of* $\mathbf{MGL}_S$. One deduces from the above lemma a canonical map:

$$c^{\mathbf{MGL}} : \Sigma^\infty \mathbb{P}^\infty_S \to \Sigma^\infty \mathrm{Th}(\gamma_1) \to \mathbf{MGL}_S(1)[2].$$

By construction (see the above proof for $n = 1$), the restriction of this map to $(\mathbb{P}^1_S, \infty)$ corresponds up to $\mathbb{P}^1$-desuspension to the unit:

$$\Sigma^\infty S_+ = \Sigma^\infty \mathrm{Th}(\gamma_0) \to \mathbf{MGL}_S$$

Therefore, $c^{\mathbf{MGL}}$ is an orientation of $\mathbf{MGL}_S$.

The main theorem of this section is the following universality theorem:

Theorem 2.1.8 (Vezzosi, Panin-Pimenov-Röndigs, Naumann-Østvær-Spitzweck)**.** *Let $\mathbb{E}$ be a ring spectrum over S. Then the following sets are in bijective correspondence:*

(i) orientations c of $\mathbb{E}$;
(ii) morphisms of ring spectra $\varphi : \mathbf{MGL}_S \to \mathbb{E}$.

[25] In fact, Tom Bachmann and Marc Hoyois have shown in [BH21, 16.2] how to give an E_∞-ring structure on $\mathbf{MGL}_S$ using the so-called motivic J-homomorphism, which is an ∞-categorical enhancement of the stable Thom space functor.

by the map:

$$(ii) \to (i), \varphi \mapsto \varphi_*(c^{\mathbf{MGL}}) \tag{2.9}$$

where $\varphi_* : \mathbf{MGL}^{**} \to \mathbb{E}^{**}$ *is the induced map on cohomology.*

In other words, $\mathbf{MGL}_S$ is the universal oriented ring spectrum over S.

Idea of Proof. There are several steps for this proof, which works as in topology.

The first step is to determine the $\mathbb{E}$-cohomology of $\mathrm{BGL}_\infty = \operatorname{colim}_n \mathrm{BGL}_n$, for $(\mathbb{E}, c)$ oriented. As in topology, one gets:

$$\mathbb{E}^{**}(\mathrm{BGL}_\infty) \simeq \mathbb{E}^{**}(S)[[c_1, c_2, ...]]$$

where c_n is the n-th Chern class of the tautological rang n bundle on BGL_n. Note in passing that the preceding computation uses the Milnor exact sequence:

$$0 \to \lim_{n\geq 0}{}^1 \mathbb{E}^{**}(\mathrm{BGL}_n) \to \mathbb{E}^{**}(\mathrm{BGL}_\infty) \to \lim_{n\geq 0} \mathbb{E}^{**}(\mathrm{BGL}_n) \to 0$$

and the vanishing of the first term (as in Remark 1.2.5).

One deduces, from a similar Milnor exact sequence and formula (2.8) that the following canonical maps are isomorphism:

$$\begin{aligned}\mathbb{E}^{**}(\mathbf{MGL}_S) \to & \lim_{n\geq 0} \mathbb{E}^{*+2n,*+n}(\mathrm{Th}(\gamma_n)) \\ & \xrightarrow{\mathrm{th}} \lim_{n\geq 0} \mathbb{E}^{**}(\mathrm{BGL}_n).\,\mathrm{th}(\gamma_n) \simeq \mathbb{E}^{**}(\mathrm{BGL}_\infty),\end{aligned}$$

where th is given by the Thom isomorphisms of γ_n constructed previously.

It follows that the sequence $(\mathrm{th}(\gamma_0), \mathrm{th}(\gamma_1), ...)$ uniquely defines an element $\mathrm{th}^c \in \mathbb{E}^{00}(\mathbf{MGL}_S)$, canonically associated to the orientation c. It remains to prove that $\mathrm{th}_c : \mathbf{MGL}_S \to \mathbb{E}$ is in fact a morphism of ring spectra. And moreover, the application $c \mapsto \mathrm{th}^c$ is a left and right inverse to (2.9). □

Remark 2.1.9. This result was first proved over a field by Vezzosi in [Vez01]. It was later revisited in [PPR08], still over a field. Both proofs were in fact valid over a general base. A definitive refrence, valid over an arbitrary base was finally given in [NSØ09], partly based on the Landweber exactness theorem.

2.1.c. **Virtual Thom classes.**

2.1.10. The Thom classes associated with on oriented ring spectrum $(\mathbb{E}, c)$ are multiplicative. Let $p : X \to S$ be a smooth morphism and consider an exact sequence of vector bundles over X:

$$0 \to V' \to V \to V'' \to 0. \tag{2.10}$$

One deduces from this exact sequence a canonical isomorphism in $\mathrm{SH}(X)$:

$$\Sigma^\infty \mathrm{Th}(V) \simeq \Sigma^\infty \mathrm{Th}(V') \otimes_X \Sigma^\infty \mathrm{Th}(V''). \tag{2.11}$$

Further:

Lemma 2.1.11. *Consider the above assumptions. Then, the isomorphism* (2.11) *induces an identification:*

$$\mathbb{E}^{**}(\mathrm{Th}(V)) \simeq \mathbb{E}^{**}(\mathrm{Th}(V')) \otimes_{\mathbb{E}^{**}(X)} \mathbb{E}^{**}(\mathrm{Th}(V'')),$$

and through this identification, one has: $\mathrm{th}(V) = \mathrm{th}(V') \otimes \mathrm{th}(V'')$.

Proof. One reduces to the case where the sequence is split. Then we have an isomorphism of vector bundles over $\mathbb{P}(V)$:

$$\xi = p^{-1}(V)/\mathcal{O}(-1) \simeq p^{-1}(V')/\mathcal{O}(-1) \oplus p^{-1}(V'')/\mathcal{O}(-1) = \xi' \oplus \xi''.$$

According to the Whitney sum formula, one deduces $e(\xi) = e(\xi').e(\xi'')$. One concludes using the fact $\mathbb{P}(V) - \mathbb{P}(V')$ is a vector bundle over $\mathbb{P}(V'')$. □

2.1.12. We can elaborate on the previous result as follows. Consider the previous notation.

We let $\underline{\mathrm{K}}(X)$ be the Picard 1-category of virtual vector bundle.[26] One deduces from the isomorphisms of the form (2.11) that the Thom spectrum functors extends to a *virtual Thom spectrum functor*[27]:

$$\underline{\mathrm{K}}(X) \to \mathrm{Ho}\,\mathrm{SH}(X), v \mapsto \mathrm{Th}(v).$$

The cohomology with coefficients in $\mathbb{E}$ can be defined on these virtual Thom spectra:

$$\mathbb{E}^{n,i}(\mathrm{Th}(v)) = [\mathrm{Th}(v), \mathbb{E}_X(i)[n]]_X.$$

There is an obvious $\mathbb{E}^{**}(X)$-module structure on these groups. Moreover, the preceding lemma shows that these $\mathbb{E}^{**}(X)$-modules are free of

[26] This is the groupoid associated with Quillen K-theory space $\mathrm{K}(X)$. A direct construction is given in [Del87].

[27] One can deduce it by elementary means, but we now know that this is actually the truncation of the (already mentioned) motivic J-homomorphism defined by Tom Bachmann and Marc Hoyois: $J : \mathrm{K}(X) \to \mathrm{Pic}(\mathrm{SH}(X))$. See [BH21, §16.2].

rank 1, and moreover, admits a canonical $\mathbb{E}^{**}(X)$-basis $\mathrm{th}(v)$ of bidegree $(2r, r)$, where $r \in \mathbb{Z}$ is the rank of v.[28]

Definition 2.1.13. Given a virtual vector bundle v over a smooth S-scheme X, of virtual rank $r \in \mathbb{Z}$, we let $\mathrm{th}(v) \in \mathbb{E}^{2r,r}(\mathrm{Th}(v))$ be the class defined above.

Bydefinition, the following morphism:

$$\mathbb{E}^{**}(X) \to \mathbb{E}^{**}(\mathrm{Th}(v)), \alpha \mapsto \alpha.\,\mathrm{th}(v).$$

is an isomorphism of $\mathbb{E}^{**}(X)$-modules, called the Thom isomorphism of v associated with the orientation c on $\mathbb{E}$.

Note finally that, as in 2.1.4, one deduces a isomorphism of $\mathbb{E}_X$-modules:

$$\mathrm{th}(v) : \mathbb{E}_X \otimes \mathrm{Th}(v) \xrightarrow{\sim} \mathbb{E}_X(r)[2r]. \tag{2.12}$$

2.2. Formal group laws and orientations

2.2.a. **Recall on formal group laws.**

2.2.1. From the point of view of algebraic geometry, a *commutative formal group law of dimension* 1 over a ring R is an *abelian group object structure* on the formal scheme $\mathrm{Spf}(R[[x]])$, in the category of formal schemes.[29] We will say FGL (over R) for abelian formal group law of dimension 1.

This is equivalent to the (usual) concrete definition: such an FGL is given by a power series $F(x, y) \in R[[x, y]]$ satisfying the properties:

(1) *Neutral element.* $F(x, 0) = x$.
(2) *Commutativity.* $F(x, y) = F(y, x)$.
(3) *Associativity.* $F(x, F(y, z)) = (F(x, F(y, z)))$.[30]

(Recall that the existence of the formal inverse follows from these conditions; see e.g. [Str19, Lem. 2.7].) We will consider the following generic form for such an FGL:

$$F(x, y) = x + y + \sum_{i,j} a_{ij}.x^i y^j.$$

[28]More precisely, one uses the universal property of the Picard category $\underline{\mathrm{K}}(X)$ and the preceding lemma to get both results.

[29]We see $R[[x]]$ as an admissible ring via the ideal of definition (x). Therefore $\mathrm{Spf}(R[[x]])$ is nothing else than the topological space $\mathrm{Spec}(R)$ seen as a ring space via the sheaf associated to the pro-ring $R[[x]]$.

[30]Note that the substitutions are licite because property (1) implies that $F(x, y)$ has no constant term.

Example 2.2.2. Recall that the only examples of formal group laws such that $F(x,y)$ has only finitely many coefficients are:

(1) *Additive FGL*: $F_{add}(x,y) = x + y$.
(2) *u-Multiplicative FGL*: $F_{mul,u}(x,y) = x + y + u.xy$, for $u \in R$.

Remark 2.2.3. FGL can be based changed: given a morphism of rings $\varphi : R \to R'$, and an FGL $F(x,y)$ over R, we obtain an FGL $F_{R'}(x,y)$ over R' by applying φ coefficient-wise.

2.2.4. Let R be a ring, and $F(x,y)$, $G(x,y)$ two FGL with coefficients in R. An isomorphism θ from $F(x,y)$ to $G(x,y)$ will be a power series $\theta(t) = a_1.t + a_2.t^2 + \ldots$ such that $a_1 \in R^\times$, and the following relation holds in $R[[x,y]]$:

$$\theta(F(x,y)) = G(\theta(x), \theta(y)).$$

When one considers $(R, F(x,y))$ and $(R', G(x,y))$ two FGL with different ring of coefficients, one defines a morphism from the first to the second as a pair (φ, θ) such that $\varphi : R \to R'$ is a morphism of rings, and $\theta : F_{R'}(x,y) \to G(x,y)$ is an isomorphism of FGL over R'. One deduces a fibered category $\mathscr{FGL}$ over rings whose objects are pairs $(R, F(x,y))$ and morphisms are described as above.

Let us recall that the theorem of Lazard asserts that $\mathscr{FGL}$ admits an initial object $(\mathbb{L}, F_{\mathbb{L}})$ such that $\mathbb{L} = \mathbb{Z}[a_1, a_2, ...]$ is a polynomial ring, now called the *Lazard ring*.

2.2.b. **Orientations and FGL.**

2.2.5. We fix an oriented motivic ring spectrum $(\mathbb{E}, c)$ over S (Definition 1.2.1), with ring of coefficients $\mathbb{E}^{**} = \mathbb{E}^{**}(S)$.

As the group scheme $\mathbb{G}_m$ is abelian, one immediately deduces that $B\mathbb{G}_m$ is an h-group *i.e.* a group object in the homotopy category of $\mathrm{Sh}^\infty(\mathscr{S}m_S)$, and therefore also after applying the $\mathbb{A}^1$-localization functor.

On the other hand $\mathbb{P}^\infty_S$, being the moduli space of line bundles (relative to S), it automatically acquires a structure of an abelian group object in the category of ind-smooth S-schemes, corresponding to the existence of tensor product of line bundles, as well as the inverse functor $\lambda \mapsto \lambda^\vee$. In particular, we get a canonical map:

$$\sigma : \mathbb{P}^\infty_S \times_S \mathbb{P}^\infty_S \to \mathbb{P}^\infty_S$$

The reader can check that it is actually given by the classical Segre embedding.

Consider an oriented ring spectrum $(\mathbb{E}, c)$. The projective bundle formula Theorem 1.2.4 implies that $\mathbb{E}^{**}$ satisfies the Künneth formula on product of projective spaces. This, together with the preceding observation, readily implies that $\mathbb{E}^{**}(\mathbb{P}^\infty_S)$ is an abelian co-group object. In other words, abelian group structure on $\mathbb{P}^\infty_S$ determines an FGL with coefficients in $\mathbb{E}^{**}(S)$: one has a canonical map:

$$\mathbb{E}^{**}[[c]] \simeq \mathbb{E}^{**}(\mathbb{P}^\infty_S) \xrightarrow{\sigma^*} \mathbb{E}^{**}(\mathbb{P}^\infty_S \times_S \mathbb{P}^\infty_S) \simeq \mathbb{E}^{**}[[x, y]]$$

and one defines $F_c(x, y)$ as the image of c.

Definition 2.2.6. Given the above notation, we will say that $F_c(x, y)$ is the *formal group law* associated with the oriented ring spectrum $(\mathbb{E}, c)$.

We will say that $(\mathbb{E}, c)$ (or just c) is respectively *additive* or *multiplicative* (with parameter u) if $F_c(x, y)$ is the additive or (u-)multiplicative) FGL.

There is a constraint on the coefficients say a^S_{ij} of $F_c(x, y)$ coming from the fact $\mathbb{E}^{**}$ is bigraded: in fact a^S_{ij} has bidegree $(2-2i-2j, 1-i-j)$ in $\mathbb{E}^{**}$.

Note that by base change along $p^* : \mathbb{E}^{**}(X) \to \mathbb{E}^{**}(S) = \mathbb{E}^{**}$, for any smooth map $p : X \to S$, we deduce from the FGL $F_c(x, y)$ over E^{**} and FGL over $\mathbb{E}^{**}(X)$ that we will denote by $F^X_c(x, y)$ (or simply $F_c(x, y)$ when X is clear).

Before giving examples, let us explain what is the concrete signification of the above FGL in term of characteristic classes.

Proposition 2.2.7. *Consider the notations of the previous definition.*

Then for any line bundles L_1, L_2 over X, one has the following relation in $\mathbb{E}^{2,1}(X)$

$$c_1(L_1 \otimes L_2) = F^X_c\big(c_1(L_1), c_1(L_2)\big).$$

Note in particular that this relation makes sense because the classes $c_1(L)$ and $c_1(L')$ are nilpotent in $\mathbb{E}^{**}(X)$ (see 1.2.7). The proof is in fact tautological.

Example 2.2.8. Consider the absolute oriented ring spectra of Example 1.2.2:

(1) The first Chern class associated with the canonical orientation of motivic cohomology is nothing else than the canonical isomorphism:

$$\mathrm{Pic}(X) \simeq \mathrm{CH}^1(X) \simeq \mathbf{H}^{2,1}_{\mathrm{M}}(X).$$

As this is an isomorphism of groups, the associated FGL is additive, *i.e.* the motivic Eilenberd-MacLane ring spectrum $\mathbf{H}_{\mathrm{M}}\mathbb{Z}$ is additive.

(2) Similarly, for the ring spectrum $\mathbf{H}_\epsilon$ represented by a mixed Weil cohomology, the first Chern class associated with the canonical orientation is induced by the cycle class:

$$c_1^\epsilon : \mathrm{Pic}(X) \simeq \mathrm{CH}^1(X) \to \mathbf{H}_\epsilon^2(X).$$

Therefore, the oriented ring spectrum $\mathbf{H}_\epsilon$ is additive.

(3) For the orientation on the algebraic K-theory spectrum $\mathbf{KGL}_S$, it follows easily from the definition that the first Chern class of a line bundle L over a smooth S-scheme X is given by the formula:

$$c_1^{\mathbf{KGL}}(L) = \beta^{-1}.(1 - [L^\vee]).$$

One deduces that the FGL associated with $\mathbf{KGL}_S$ is the multiplicative one with parameted $-\beta$:

$$F_{\mathbf{KGL}}(x, y) = x + y - \beta.xy.$$

(4) many more example have been constructed: an algebraic version of the Brown-Peterson spetrum ([Vez01]), of the Morava K-theory[31], elliptic ring spectrum ([LYZR19]). These constructions are all based on the next theorem and on the motivic version of the Landweber exactness theorem [NSØ09].

2.2.c. **Algebraic cobordism and the Lazard ring.** We end up this course with a discussion of the motivic analog of Quillen's theorem on complex cobordism and the Lazard ring.

Let us recall that Lazard has proved that there is a universal formal group law $(\mathbb{L}, F_\mathbb{L})$, whose coefficient ring $\mathbb{L} = \mathbb{Z}[b_1, b_2, ...]$ is a polynomial algebra over $\mathbb{Z}$ with infinitely many variables. In particular, for any scheme S, the canonical orientation on $\mathbf{MGL}_S$ gives rise to a canonical map

$$\varphi_S : \mathbb{L} \to \mathbf{MGL}^{2*,*}(S)$$

where the right hand-side is the graded part $(2n, n)$ of the algebraic cobordism ring of S. In his ICM talk in 1998, Voevodsky made the following conjecture (see [Voe98]):

Conjecture. *For any regular local scheme S, the map φ_S is an isomorphism.*

Here is the current best result on this conjecture.

Theorem 2.2.9 (Levine, Hoyois, Spitzweck)**.** *Let S be a local prosmooth scheme over a field of characteristic exponent e or a DVR of mixed characteristic* $(0, e)$.

[31] initially proposed by Voevodsky in a 1995 preprint

Then the map $\varphi_S[1/e]$ *is an isomorphism, and the algebraic cobordism* $\mathbf{MGL}_S[1/e]$ *as the universal formal group law.*

In fact, Levine was the first to give the proof, in [Lev09], of the above theorem when S is the spectrum of a field k of characteristic 0. It was in fact a corollary of the isomorphism between the geometric part of the cohomology represented by $\mathbf{MGL}_k$ and the "concrete" algebraic cobordism theory, defined by explicit generators and relations, by Levine and Morel ([LM07]). Indeed, Levine and Morel had proved that the analogous of the above theorem for their cobordism theory in characteristic 0.

Hoyois ([Hoy15]) proved the above theorem directly when S is a field of characteristic exponent $e = p$, by coming back to the strategy of Quillen, using the following fundamental relation between algebraic cobordism and the motivic Eilenberg-MacLane spectrum: under the above assumption, the canonical morphism of motivic spectra

$$\mathbf{MGL}_S/(b_1, b_2, ...) \to \mathbf{H}_{\mathrm{M}}\mathbb{Z}_S$$

is an equivalence after inverting e. The global strategy as well as the above isomorphism were devised by Hopkins and Morel (unpublished).

Then Spitzweck proved the theorem in the case stated above in [Spi20], by using the preceding result, extended to a general S using his construction of the motivic Eilenberg-MacLane spectrum, and also by using the slice filtration.

Exercices

Exercice 1. Prove that infinite suspension of Thom spaces induce a functor on the groupoid of virtual vector bundles. Using the method of Riou: see 4.1 in `http://www.math.u-psud.fr/~riou/doc/operations.pdf`.

Eventually: explain the construction of Bachmann-Hoyois: [BH21, 16.1] ?

Exercice 2 (FGL). Prove the existence and unicity of the exponential strict isomorphism associated to a rational FGL. [Str19]

Show the existence of the Lazard ring.

Show Lazard theorem. (cf. `https://people.math.harvard.edu/~lurie/252xnotes/Lecture2.pdf`, +lecture3,4) ?

Exercice 3. Compute Chern classes of a tensor product for multiplicative FGL.

Cours 3. Fundamental classes and Grothendieck-Riemann-Roch theorems

This course will focus on techniques to build the so-called Gysin maps in motivic homotopy theory, *i.e.* exceptional functoriality. It is based on a joint work with Fangzhou Jin and Adeel Khan: [DJK21] (see in particular Theorem 3.1.8).

3.1. Bivariant theory and fundamental classes

3.1.a. **Overview of Grothendieck 6 functors formalism.** The following theorem sumarize the core properties of the Grothendieck 6 functors formalism for the motivic stable homotopy category. It is the exact analog of the formalism satisfied by torsion (resp. ℓ-adic) étale sheaves proved in SGA4, except one has to consider Thom spaces as natural twists in the smooth purity property. It is mainly due to Voevodsky and Ayoub, with later improvements from various authors.[32] The first complete proof was given in Ayoub's PhD [Ayo07].

Theorem 3.1.1 (Voevodsky, Ayoub)**.** *The motivic stable homotopy category* $\mathrm{SH}(S)$ *for various schemes* S *is endowed with six* ∞*-functors:*

- *the adjoint pair* $(\otimes_S, \underline{\mathrm{Hom}}_S)$ *which comes from the (presentable closed symmetric) monoidal* ∞*-category* $\mathrm{SH}(S)$*;*
- *For any morphism* $f : T \to S$ *of schemes, an adjoint pair of* ∞*-functors:*
$$f^* : \mathrm{SH}(S) \rightleftarrows \mathrm{SH}(T) : f_*$$
which actually comes from an ∞*-fonctor:* $\mathrm{SH}^* : \mathscr{S}ch^{op} \to Cat_\infty^\otimes$*.*
- *for any separated*[33] *morphism of finite type* $p : Y \to X$ *in* $\mathscr{S}ch$*, a pair of adjoint functors*
$$p_! : \mathrm{SH}(Y) \rightleftarrows \mathrm{SH}(X) : p^!$$
which comes from an ∞*-functor:* $\mathrm{SH}_! : \mathscr{S}ch \to Cat_\infty$*.*

These functors satisfy the following properties:

(1) Proper support*. There exists a natural transformation of* ∞*-functors* $\mathrm{SH}_! \to \mathrm{SH}_*$ *such that the corresponding map* $\alpha_p : p_! \to p_*$ *is an isomorphism whenever* p *is proper.*

[32] First to avoid quasi-projectivity assumptions for the adjoint pair $(f_!, f^!)$, second to promote these to a pair of adjoint ∞-functors.

[33] This assumption can be removed using Zariski descent.

(2) Smooth purity. *For any smooth separated morphism $p : X \to S$ with tangent bundle T_f, there is a canonical isomorphism*

$$\mathfrak{p}_f : f^* \longrightarrow \mathrm{Th}(-T_f) \otimes f^!$$

(3) Base change. *For any cartesian square:*

$$\begin{array}{ccc} Y' & \xrightarrow{p'} & X' \\ {\scriptstyle f'}\downarrow & \Delta & \downarrow{\scriptstyle f} \\ Y & \xrightarrow[p]{} & X, \end{array}$$

such that p is separated of finite type, there exist natural isomorphisms

$$f^* p_! \xrightarrow{\sim} p'_! f'^* \,,$$
$$p^! f_* \xrightarrow{\sim} f'_* p'^! \,.$$

(4) Projection formulas. *For any separated morphism of finite type $f : Y \to X$, there exist natural isomorphisms*

$$Ex(f_!^*, \otimes) : (f_! K) \otimes_X L \xrightarrow{\sim} f_!(K \otimes_Y f^* L)\,,$$
$$\underline{\mathrm{Hom}}_X(f_!(L), K) \xrightarrow{\sim} f_* \underline{\mathrm{Hom}}_Y(L, f^!(K))\,,$$
$$f^! \underline{\mathrm{Hom}}_X(L, M) \xrightarrow{\sim} \underline{\mathrm{Hom}}_Y(f^*(L), f^!(M))\,.$$

(5) Localization. *For any closed immersion $i : Z \to S$ with complementary open immersion j, there exists distinguished triangles of natural transformations as follows:*

$$j_! j^! \xrightarrow{\alpha'_j} 1 \xrightarrow{\alpha_i} i_* i^* \xrightarrow{\partial_i} j_! j^! [1]$$
$$i_! i^! \xrightarrow{\alpha'_i} 1 \xrightarrow{\alpha_j} j_* j^* \xrightarrow{\tilde{\partial}_i} i_! i^! [1]$$

where $\alpha'_?$ (resp. $\alpha_?$) denotes the counit (resp. unit) of the relevant adjunction.

Actually the method of proof (slightly revisited in [CD19]) consists first to prove the localization property. One deduces the construction of $(f_!, f^!)$ using a method due to Deligne. Then, given the definition of the motivic stable homotopy category from the smooth site $\mathscr{S}m_S$ for the Nisnevich topology, one formally obtains that p^* admits a left adjoint $p_\sharp$ when p is smooth. Then we prove the smooth purity formula in the adjoint form: $p_! \simeq p_\sharp(\mathrm{Th}(T_p) \otimes -)$. This allows to deduce all the remaining properties, from the analogous properties of $p_\sharp$ (or by adjunction).

Example 3.1.2. Let $p : X \to S$ be a smooth (seperated) morphism. One deduces from the above sketch of proof that:

$$\Sigma^\infty X_+ \simeq p_! p^! (\mathbb{1}_S).$$

In this sense, $\Sigma^\infty X_+$ can be seen as the homotopy type of X over S, analog to the homological motives.

The above formalism is very strong. It is a nice exercice to deduce (from (1), (2), (4)) that whenever p is smooth and proper, the object $\Sigma^\infty X_+$ is strongly dualizable (aka rigid) with dual $\Sigma^\infty \operatorname{Th}(-T_f)$.

3.1.b. **Bivariant theories with twists.** One of the nice application of the six functors formalism is that it allows to associated several homology/cohomology theories with respect to a ring spectrum $\mathbb{E}$. The following one will be crucial for the remaining of the course.

Definition 3.1.3. Consider a ring spectrum $\mathbb{E}$ over a scheme S. Let $p : X \to S$ be a separated morphism of finite type, v a virtual bundle over X, and $n \in \mathbb{Z}$ be an integer. One defines the *bivariant* $\mathbb{E}$-*theory/homology* (aka *Borel-Moore* $\mathbb{E}$-*homology*) of X/S in degree n and twist v as:

$$\mathbb{E}_n(X/S, v) = [\operatorname{Th}(v)[n], f^! \mathbb{E}]$$

where the stable Thom space $\operatorname{Th}(v)$ was defined in the preceding course (see 2.1.12).

Note that cohomology is in fact a particular case of bivariant theory, according to the relation:

$$\mathbb{E}_n(X/X, v) = \mathbb{E}^{-n}(\operatorname{Th}(v)).$$

Example 3.1.4.

(1) If $\mathbb{E}$ is the ring spectrum representing Betti cohomology with integral coefficients, $v = 0$ and $S = \operatorname{Spec}(k)$, then the above bivariant theory is nothing else than the classical Borel-Moore homology of X.

(2) Let $f = i : Z \to X$ be a closed immersion, X being smooth over S. Then it follows from the localization property for the immersion i that

$$\mathbb{E}_n(Z/X, 0) \simeq [\Sigma^\infty X/X - Z, \mathbb{E}[-n]] = \mathbb{E}_Z^{-n,0}(X).$$

In other words, the bivariant $\mathbb{E}$-homology of Z/X in degree n with the $\mathbb{E}$-cohomology of X with support in Z in degree $(-n)$.

(3) if S is regular, it follows from a theorem of Jin [Jin19] that

$$\mathbf{KGL}_n(X/S) \simeq K'_n(X)$$

where left hand-side is Quillen K'-theory of X (deduced from the exact category of coherent sheaves over X).

3.1.5. The above bivariant theories satisfy good properties, which were axiomatically described by Fulton and MacPherson [FM81] except for the twists (see also [BGI71]).

(1) *Base change.* for any map $f : T \to S$, one gets a base change map: $f^* : \mathbb{E}_n(X/S, v) \to \mathbb{E}_n(X \times_S T/T, g^{-1}v)$.
(2) *Proper covariance.* For any proper morphism $f : X \to Y$ of s-schemes over S, there is a direct image map $f_* : \mathbb{E}_n(X/S, f^*v) \to \mathbb{E}_n(Y/S, v)$.
(3) *Étale contravariance.* For any étale $f : X \to Y$ of s-schemes over S, there is an inverse image map $f^! : \mathbb{E}_n(Y/S, v) \to \mathbb{E}_n(X/S, f^*v)$.
(4) *Product.* For s-morphisms $p : X \to S$ and $q : Y \to X$, and any virtual bundles $v \in \underline{\mathrm{K}}(X)$, $w \in \underline{\mathrm{K}}(Y)$, there is a product map:

$$\mathbb{E}_n(Y/X, w) \otimes \mathbb{E}_m(X/S, v) \to \mathbb{E}_{n+m}(Y/S, w + q^{-1}v).$$

These structures satisfy all the properties stated by Fulton and MacPherson (functoriality, base change formula both with respect to base change and étale contravariance, compatibility with pullbacks and projection formulas). We refer to [Dég18a, 1.2.8] and [DJK21, §2.2].

Remark 3.1.6. In case $\mathbb{E} = \mathbb{1}_S$, we simply put:

$$H_n^{\mathbb{A}^1}(X/S, v) = [\Sigma^\infty \operatorname{Th}(v)[n], f^!\mathbb{1}_S].$$

This will be called the $\mathbb{A}^1$-bivariant theory. It is universal in the sense that given any ring spectrum $\mathbb{E}$, the unit map induces canonical map:

$$H_n^{\mathbb{A}^1}(X/S, v) \to \mathbb{E}_n(X/S, v)$$

which can be seen as a (homological or bivariant) generalized regulator.[34] The above map is obviously compatible with the functorialities and the product described above.

3.1.c. **Construction of fundamental classes.**

3.1.7. A morphism $f : Y \to X$ of schemes will be called smoothable lci if it admits a factorisation $Y \xrightarrow{i} P \xrightarrow{p} X$ such that p is smooth and i is a regular closed immersion.

For such a morphism, one defines the *virtual tangent bundle* as the virtual bundle on Y equals to:

$$\tau_f = [i^{-1}T_p] - [N_i].$$

[34] This would exactly be a bivariant regulator if we were working with the ∞-category DM of mixed motivic complexes.

Alternatively, it is the virtual bundle associated with the cotangent complex $\mathcal{L}_f$ of f.[35] Virtual tangent bundles enjoy good properties. First, they are stable under pullback along a transversal map. Second, for a commutative diagram of smoothable lci morphisms:

$$\begin{array}{ccccc} Z & & \xrightarrow{\quad h \quad} & & X \\ & {}_g\searrow & & \nearrow_f & \\ & & Y & & \end{array} \tag{3.13}$$

one has $\tau_h = \tau_g + g^{-1}\tau_f$ in $\underline{\mathrm{K}}(Z)$.

The main result we will be using to get Gysin maps is the following one, proved in [DJK21, Th. 3.3.2]:

Theorem 3.1.8 (Jin-Khan-D.)**.** *For any smoothable lci $f : Y \to X$ with virtual tangent bundle τ_f, there exists a class:*

$$\eta_f \in \mathbb{E}_0(X, \tau_f)$$

called the (refined) fundamental class of f. *The collection formed by these classes satisfies the following properties:*

(1) $\eta_{\mathrm{Id}} = 1_E$, *the unit of the ring spectrum* $\mathbb{E}$.

(2) For any morphism $p : X' \to X$ which is transversal[36] *to f, $f' = f \times_X X'$, one has:*

$$p^*\eta_f = \eta_{f'}.$$

(3) For any diagram (3.13), *one has in* $\mathbb{E}_0(Z/X, \tau_h) \simeq \mathbb{E}_0(Z/X, \tau_g + g^{-1}\tau_f)$*:*

$$\eta_h = \eta_g.\eta_f$$

using the product defined in 3.1.5.

Idea of proof. One first restrict to the case where f is either a smooth morphism or a closed immersion. In the first case, η_f can be deduced from the purity property of the six functors formalism Theorem 3.1.1. In the second case, $f = i : Z \to X$, one uses the (affine) deformation space $D = D(X, Z) = B_Z(\mathbb{A}^1_X) - B_Z X$ to the normal cone (actually bundle as i is regular) $N = N(X, Z)$ associated with i. Recall that D is a scheme over $\mathbb{A}^1$, isomorphic to $X \times \mathbb{G}_m$ over $\mathbb{G}_m$, and to N over 0. Then η_i is the image of the unit by the following composite map:

$$\mathbb{E}_0(X/X, 0) \xrightarrow{\gamma_t} \mathbb{E}_{-1}(\mathbb{G}_m \times X/X, 0) \xrightarrow{\partial_{D,N}} \mathbb{E}_0(N/X, 0) \simeq \mathbb{E}_0(Z/X, [-N]).$$

[35] Such a virtual bundle exists as, in that case, $\mathcal{L}_f$ is a perfect complex of $\mathcal{O}_Y$-modules.

[36] *i.e.* it preserves the relative dimension: for any point $y' \in Y \times_X X'$ with image $y \in Y$, the dimension of the fiber of f' at y' is equal to the dimension of the fiber of f at y.

where γ_t is multiplication by the symbol $[t]$ already encountered, associated with the canonical unit t of $\mathbb{G}_m = \mathrm{Spec}(k[t, t^{-1}])$, $\partial_{D,N}$ is the residue associated with the closed immersion $N \subset D$ (whose complement is $D - N \simeq \mathbb{G}_m \times X$), and the last isomorphism is tautological. Essentially, we have:

$$\eta_i = \partial_{D,N}([t]).$$

The rest of the proof consists un showing that these two constructions glue correctly in order to ensure that property (2) holds. The key point is that for i a closed immersion which admits a smooth retraction p, one get: $\eta_i.\eta_p = 1$ (see [DJK21, Cor. 3.2.17]). □

Remark 3.1.9. We refer the reader to *loc. cit.* for the excess intersection formula, which generalizes the above point (1) in the non-transversal case.

Example 3.1.10. Fundamental classes induce all sorts of Gysin maps, provided one considers the right twists.

(1) When $f : X \to S$ is proper smoothable lci, on deduces a pushforward map:

$$f_! : \mathbb{E}^{**}(X, \mathrm{Th}(T_f) + f^{-1}v) \to \mathbb{E}^{**}(S), x \mapsto f_*(x.\eta_f)$$

where we have used the product and the pushforward map in bivariant $\mathbb{E}$-homology. Beware that the twist on the source as to be of the exact above form in order to get a Gysin map under this generality.

According to the above theorem, these Gysin maps satisfy the required properties: functoriality, base change formula with respect to transversal base change (and an excess intersection formula according to the previous remark). One can also prove, the projection formula with respect to products.

(2) In fact, one deduces from the case of of the $\mathbb{A}^1$-bivariant theory $H^{\mathbb{A}^1}$ that fundamental classes corresponds to the following *trace map*:

$$\mathrm{tr}_f : \mathrm{Id}_S \to f_*(\mathrm{Th}(\tau_f \otimes f^!(-))).$$

(and dually a cotrace map). This generalizes a classical construction in étale cohomology (see [AGV73, Tome III]). We refer the reader to [DJK21, §4.3] for more details.

3.1.d. **Oriented fundamental classes.**

3.1.11. Consider now an oriented ring spectrum $(\mathbb{E}, c)$. The virtual Thom isomorphism also induces Thom isomorphisms for twisted bivariant theory. Indeed, we first remark that one can vary the product in bivariant theory to get the following variant of cap-product:[37]

$$\mathbb{E}^{m,j}(\mathrm{Th}(w)) \otimes \mathbb{E}_{n,i}(X/S, v) \to \mathbb{E}_{n-m,i-j}(X/S, v+w).$$

Let v be a virtual bundle of rank $r \in \mathbb{Z}$. We have obtained in Definition 2.1.13 a class:

$$\mathrm{th}(-v) \in \mathbb{E}^{-2r,-r}(\mathrm{Th}(-v)).$$

One deduces the following Thom isomorphism for bivariant theories.

Lemma 3.1.12. *Consider the above notation. Then the following map is an isomorphism:*

$$\mathbb{E}_0(X/S, v) \to \mathbb{E}_{2r,r}(X/S), \alpha \mapsto \mathrm{th}(-v).\alpha.$$

Proof. By unfolding the definitions, one deduces that the above map is induced by the isomorphism (2.12). □

Using the preceding Thom isomorphism, one immediately deduces the following central definition for motivic oriention theory.

Definition 3.1.13. Consider the above notation. Let $f : X \to S$ be a smoothable lci morphism of relative dimension $d \in \mathbb{Z}$. Then we define the oriented fundamental class of f with coefficients in $(\mathbb{E}, c)$ as:

$$\eta_i^c := \mathrm{th}(-\tau_f).\eta_f \in \mathbb{E}_{2d,d}(X/S).$$

When f is in addition proper, one deduces a *Gysin morphism without twists*:

$$f_! : \mathbb{E}^{**}(X) \to \mathbb{E}^{*+2d,*+d}(S), \alpha \mapsto f_*(\alpha.\eta_f^c).$$

In fact, more directly and dually, one gets *Gysin morphism* in bivariant homology. Let $f : Y \to X$ be a smoothable lci morphism between s-schemes over S. Then one gets:

$$f^! : \mathbb{E}^{**}(X/S) \to \mathbb{E}^{*+2d,*+d}(Y/S), \alpha \mapsto \eta_f^c.\alpha.$$

Remark 3.1.14. (1) Given the properties of the twisted fundamental classes stated above, and the one of Thom isomorphisms, this Gysin maps satisfy all the desired properties: transversal base change, compatibility with composition, excess intersection formula. The projection formula follows from the transversal base change one.

[37] This is obtained via tensor product of maps in $\mathrm{SH}(X)$, using the pairing $f^* \otimes f^! \to f^!$. The pairing is a consequence of the projection formula: indeed one gets a map:

$$f_!(f^*A \otimes f^!B) \xrightarrow{\sim} A \otimes f_!f^!(B) \xrightarrow{Id_A \otimes \alpha_f} A \otimes B$$

using the projection formula and the unit of the adjunction $(f_!, f^!)$. The required map follows by adjunction.

(2) As in the case of "twisted" fundamental classes, one can associate to the oriented fundamental classes many more Gysin-type morphisms. See e.g. [Dég18a].

Example 3.1.15. (1) Let $i : Z \to X$ be a regular closed immersion of codimension n. Then $\eta_i^c \in \mathbb{E}_{-2n,-n}(Z/X) = \mathbb{E}_Z^{2n,n}(X)$ is a refinment of the classical fundamental class of Z, the supported version.

In motivic cohomology, at least when X is smooth over a regular base S of dimension less than 1, the class $i_*(1) = \mathcal{O}(\eta_i^c)$ does corresponds to the class $[Z]$ in $\mathrm{CH}^n(X)$, after forgetting the support. Similarly, the Gysin map when f is proper between smooth S-schemes does corresponds to the usual pushforward of cycles.

In K-theory, when X is regular and c is the orientation previously defined, η_i^c corresponds to the element in $\mathrm{K}^Z(X)$ corresponding to $[\mathcal{O}_Z]$ in K'-theory (using that is X regular to deduce an element in K-theory with support).

(2) Let $f : X \to \mathrm{Spec}(k)$ be a smooth proper morphism of dimension d over a field of characteristic 0. Then the element $f_*(1) = \mathcal{O}(\eta_f^c)$ in $\mathbf{MGL}_{2d,d}(X)$ does corresponds to the cobordism class $[X] \in \Omega^d(X)$, via the isomorphism of Theorem 2.2.9.

Let us end-up these examples with the following computation obtained in [Dég18b, Ex. 3.2.14].

Theorem 3.1.16. *Let $(\mathbb{E}, c)$ be an oriented ring spectrum over S with associated FGL $F_c(x, y) = \sum_{i,j} a_{ij}.x^i y^j$. Then one has the following equality in $\mathbb{E}_{2n,n}(S)$:*

$$[\mathbb{P}^n_S]_{\mathbb{E}} = p_*^c(1) = (-1)^{\lfloor (n+1)/2 \rfloor}.\begin{vmatrix} 0 & \cdots & 0 & 1 & a_{1,1} \\ \vdots & \cdot^{\cdot^{\cdot}} & & & a_{1,2} \\ 0 & & & & \vdots \\ 1 & & & & \\ a_{1,1} & a_{1,2} & \cdots & & a_{1,n} \end{vmatrix}.$$

In the particular case where the oriented ring spectrum $\mathbb{E}$ represents the complex cobordism theory $\mathrm{MU}^*(X(\mathbb{C}))$ for complex varieties X, the above theorem is a determinantal formula of the celebrated Myschenko formula, which computes the cobordism class of $\mathbb{CP}^n$.

3.2. GRR THEOREMS

3.2.a. **Morphisms of ring spectra.**

3.2.1. Consider oriented motivic ring spectra $(\mathbb{E}, c)$ and $(\mathbb{F}, d)$ over S, with respective formal group laws $F_{\mathbb{E}}(x, y)$ and $F_{\mathbb{F}}(x, y)$. Let $\phi : \mathbb{E} \to \mathbb{F}$ be a morphism of ring spectra. It induces a morphism in cohomology, and in particular a map:

$$\mathbb{E}^{**}[[c]] \simeq \mathbb{E}^{**}(\mathbb{P}^\infty_S) \xrightarrow{\phi_*} \mathbb{F}^{**}(\mathbb{P}^\infty_S) \simeq \mathbb{F}^{**}[[d]]$$

In particular, on can write $\phi_*(c) = \theta_\phi(d)$ where $\theta_\phi(t) \in \mathbb{F}^{**}[[t]]$ is a power series. As ϕ_* is a ring morphism, one immediately deduces that $\phi_*(d)$ is an orientation of $\mathbb{F}$, and θ_ϕ has the following form:

$$\theta_\phi(t) = t + \dots.$$

Finally, one deduces from the definition of the associated FGL the following relation:

$$\theta_\phi(F_E(x, y)|_{F^{**}}) = F_{\mathbb{F}}(\theta_\phi(x), \theta_\phi(y)).$$

In other words, θ_ϕ is a *strict isomorphism* from the FGL $F_{\mathbb{E}}(x, y)|_{\mathbb{F}^{**}}$ extended along $\phi_* : \mathbb{E}^{**} \to \mathbb{F}^{**}$ to the FGL $F_{\mathbb{F}}(x, y)$.

Definition 3.2.2. In the above situation, we will say that $(\phi_*, \theta_\phi) : (\mathbb{E}^{**}, F_{\mathbb{E}}) \to (\mathbb{F}^{**}, F_{\mathbb{F}})$ is the morphism of FGL associated with the ring morphism ϕ.

This construction is compatible with base change along any map $p : X \to S$: using the same construction, for the oriented ring spectrum $(\mathbb{E}_X, c_X)$, we obtain a power series θ^X_ϕ as above. Then $\theta^X_\phi = p^*(\theta_\phi) \in \mathbb{F}^{**}(X)$.

Remark 3.2.3. Note that we can take $\phi = \mathrm{Id}_{\mathbb{E}}$, but still considering two orientations c, d on $\mathbb{E}$. The situation corresponds to a change of orientation, and in this case we denote by $\theta_{c,d}(t)$ the power series obtained from the above construction. It is a strict isomorphism of FGL: $\theta_{c,d} : F_c(x, y) \to F_d(x, y)$ with coefficients in $\mathbb{E}^{**}$.

In fact, the formulas below will describe how the invariants of an oriented ring spectrum change when changing the orientation.

Example 3.2.4. (1) The most important example for us will be the *Chern character*. Over any base scheme S, it is an isomorphism of ring spectra of the form:

$$\mathrm{ch} : \mathbf{KGL}_S \otimes \mathbb{Q} \to \bigoplus_{n \in \mathbb{Z}} \mathbf{H}_{\mathrm{M}}\mathbb{Q}_S(n)[2n]$$

This isomorphism corresponds to the decomposition of rational (homotopy invariant) K-theory with respect to eigenspaces for

Adams operations, or equivalently in the graded parts associated with the γ-filtration. Under this form, it is essentially due to Riou; see [CD19, Lem. 14.1.4].

The Chern character is essentially determined by its value on $\mathbf{KGL}^{0,0} = K_0$ on line bundles L over say a smooth S-scheme X:

$$\mathrm{ch}([L]) = \sum_{n \geq 0} \frac{1}{n!}.c_1(L)^i.$$

As the FGL on $\mathbf{H}_{\mathrm{M}}\mathbb{Q}_S$ is additive, and that on $\mathbf{KGL}_S$ is multiplicative, the isomorphism of FGL associated with ch is necessarily the unique strict isomorphism given by the *exponential*.

(2) It was proved in [CD19] that $\mathbf{H}_{\mathrm{M}}\mathbb{Q}_S$ is the universal orientable rational ring spectrum. In other words, any orientable rational ring spectrum $\mathbb{E}$ inherits a canonical ring map $\phi_{\mathbb{E}} : \mathbf{H}_{\mathrm{M}}\mathbb{Q}_S \to \mathbb{E}$. If one chooses an orientation c on $\mathbb{E}$, then the morphism of formal group law associated with $\phi_{\mathbb{E}}$ will be logarithm of the FGL associated with c.

(3) Any stable cohomological operation on an oriented ring spectrum will give rise to interesting examples. A particular interest has been put on the Adams operations (see [Sou85]).

(4) It is worth mentioning that the morphism of ring spectra $\phi : \mathbf{MGL}_S \to \mathbb{E}$ that arises by universality of algebraic cobordism (Theorem 2.2.9) does preserves the orientation, by the very construction. In other words, the associated strict isomorphism is just the identity: $\theta_\phi(t) = t$.

3.2.b. **Todd classes.**

Proposition 3.2.5. *Consider oriented motivic ring spectra $(\mathbb{E}, c)$ and $(\mathbb{F}, d)$ over S, a morphism of ring spectra $\phi : \mathbb{E} \to \mathbb{F}$ and use the notation of Definition 3.2.2.*

Then for any smooth S-scheme X, there exists a unique morphism of abelian groups:

$$\mathrm{td}_\phi : K_0(X) \to \mathbb{F}^{00}(X)^\times$$

natural in X with respect to pullbacks, and such that for a line bundle L over X, one gets[38]*:*

$$\mathrm{td}_\phi(L) = \frac{t}{\theta_\phi(t)}(t = d_1(L)).$$

[38] The right hand-side is well defined as the Chern class $d_1(L)$ is nilpotent.

Moreover, for any vector bundle V/X of rank n, one has the relation between the respective Chern classes:

$$d_n(V) = \mathrm{td}_\phi(V).\phi_*(c_n(V)).$$

Remark 3.2.6. Taking $\phi = \mathrm{Id}_{\mathbb{E}}$, the above formula allows to express Chern classes associated with the orientation d in terms of the Todd classes in associated with c and Todd classes. Using the splitting principle, one can find an expression for all Chern classes, generalizing the last formula.

3.2.c. **Grothendieck-Riemann-Roch theorems.** We finally obtain a Grothendieck-Riemann-Roch formula *à la* Fulton-MacPherson:

Theorem 3.2.7. *We consider the notation of Proposition 3.2.5. Let $f : X \to S$ be a smoothable lci morphism of dimension n, with virtual bundle τ_f, $\eta_f^c \in \mathbb{E}_{2n,n}(X/S)$ and $\eta_f^d \in \mathbb{F}_{2n,n}(X/S)$ be the respective associated oriented fundamental classes.*

Then, the following formula holds in $\mathbb{F}_{2n,n}(X/S)$:

$$\phi_*(\eta_f^c) = \mathrm{td}_\phi(\tau_f).\eta_f^d.$$

Proof. Given the (formidable) theoretical background used, the proof is now very easy. First, one comes back to the definition of oriented fundamental classes:

$$\phi_*(\eta_f^c) = \phi_*(\mathrm{th}^c(-\tau_f).\eta_f^{\mathbb{E}}) = \phi_*(\mathrm{th}^c(-\tau_f)).\eta_f^{\mathbb{F}}. \tag{3.14}$$

The last equality follows by definition of fundamental classes. Then we are reduced to prove the relation for a virtual bundle v over X:

$$\phi_*(\mathrm{th}^c(v)) = \mathrm{td}_\phi(-v).\,\mathrm{th}^d(v).$$

But by the splitting principle, one reduces to the case $v = [L]$, which follows by the very construction of todd classes and the relation $\mathrm{th}(L) = c_1(L)$. □

This result implies several variants of the Grothendieck-Riemann-Roch formula, depending if you consider cohomology, compactly supported cohomology, Borel-Moore homology. Let us mention the following interesting formula (see [Dég18a, 3.3.12] in the case of a field; the case of a Dedekind scheme uses the localization long exact sequence on higher Chow groups, due to Marc Levine, and their representability in motivic homotopy due Markus Spitzweck: [Spi18]):

Theorem 3.2.8. *Let S be a field or a Dedekind ring.*

Let $f : Y \to X$ be a global complete intersection of S-schemes such that:

- *X and Y are separated of finite type if S is the spectrum of a field;*
- *X and Y are smooth over S, or smooth over a residue field of S if S is a Dedekind scheme.*

Then we get the following commutative diagram:

$$\begin{array}{ccc} K'_n(X) & \xrightarrow{f^!} & K'_n(Y) \\ \downarrow{\scriptstyle \mathrm{ch}_X} & & \downarrow{\scriptstyle \mathrm{ch}_Y} \\ \bigoplus_{i\in\mathbb{Z}} \mathrm{CH}_i(X,n)_{\mathbb{Q}} & \xrightarrow{\mathrm{td}(\tau_f).f^!} & \bigoplus_{i\in\mathbb{Z}} \mathrm{CH}_i(Y,n)_{\mathbb{Q}} \end{array}$$

where ch *is a Chern character isomorphism,* td *is the Todd class associated with* ch *(as a ring spectrum), and the upper (resp. lower) map $f^!$ is the Gysin morphism associated with f on Quillen's K'-theory (resp. higher Chow groups).*

Cours 4. **A quadratic Riemann-Roch theorem**

This last course is dedicated to a notion of orientation suited for describing the quadratic phenomenas arising in motivic homotopy theory. While this theory is based on the foundational work of Panin and Walter, many developments have been obtained in collaboration with Jean Fasel, and also with Jens Hornbostel and David Coulette. I will describe here some of these latest developments.

4.1. Non orientable motivic spectra

4.1.a. **Chow-Witt groups.**

4.1.1. *Stable homotopy of motivic spheres.* We have seen that the motivic stable homotopy $\mathrm{SH}(k)$ over a field admits a t-structure defined by Morel, the *homotopy t-structure.*

Let k be a perfect field (or k non perfect, but one inverts its residue characteristic). Recall that given a motivic spectrum $\mathbb{E}$, one defines its *i-th homotopy sheaves* for $i \in \mathbb{Z}$ as the graded Nisnevich sheaf $\underline{\pi}_i(\mathbb{E})_*$ associated with the presheaf

$$X \mapsto [\Sigma^\infty X_+, \mathbb{E}(n)[n-i]], n \in \mathbb{Z}.$$

Recall this object actually is a homotopy module[39] and it is completely determined by its value on function fields.[40]

[39] *i.e.* an abelian $\mathbb{Z}$-graded Nisnevich sheaf M_* on $\mathscr{S}m_k$ equiped with isomorphism $\epsilon_n : (M_{n+1})_{-1} \to M_n$.

[40] More precisely, it is equivalent to a Milnor-Witt cycle module in the sense of Feld (see [Fel20, Fel21]).

We know a lot about the homotopy sheaves of the sphere spectrum $\mathbb{1}_k = \mathbb{S}^0$. First, it is non-negative for the homotopy t-structure. Moreover, its 0-th homotopy sheaf can be computed according to the fundamental theorem of Morel:

$$\underline{\pi}_0(\mathbb{1}_k)_* \simeq \underline{\mathrm{K}}_*^{\mathrm{MW}}$$

where the right hand-side is the unramified Milnor-Witt K-theory sheaf.

Let us however mention that at least in characteristic 0, it is unbounded, and has non trivial homotopy sheaves in all non-negative degrees.[41] On the contrary, it is expected that over a field of positive characteristic, the rational sphere spectrum $\mathbb{1}_k \otimes \mathbb{Q}$ is concentrated in degree 0, isomorphic to the rational unramified Milnor K-theory sheaf.[42]

4.1.2. *Chow-Witt groups.* From the preceding paragraph, we see that the homotopy module $\underline{\mathrm{K}}_*^{\mathrm{MW}}$, as the 0-th homotopy of the sphere spectrum, is of fundamental importance. But unlike in classical homotopy theory, where the 0-th homotopy of the sphere spectrum is the Eilenberg-MacLane spectrum of $\mathbb{Z}$, the spectrum $\mathbf{H}\,\underline{\mathrm{K}}_*^{\mathrm{MW}}$ corresponding to $\underline{\mathrm{K}}_*^{\mathrm{MW}}$ is *not orientable.*

Indeed, the Hopf map η certainly acts non-trivial on $\underline{\mathrm{K}}_*^{\mathrm{MW}}$ as it is an isomorphism on the negative part: for any $n < 0$, the multiplication by η induce an isomorphism

$$\mathrm{K}_{-n}^{\mathrm{MW}}(F) \xrightarrow{\gamma_\eta} \mathrm{K}_{-n-1}^{\mathrm{MW}}(F)$$

both groups being isomorphic to the Witt group $W(F)$. One of the aim of this lecture is to explain that one can still develop an interesting (generalized) orientation theory and associated characteristic classes for $\mathbf{H}\,\underline{\mathrm{K}}_*^{\mathrm{MW}}$ (thus reconnecting to stable homotopy of topological spaces).

Recall finally that the ring spectrum $\mathbf{H}\,\underline{\mathrm{K}}_*^{\mathrm{MW}}$ represents over a field k the Chow-Witt ring:

$$[\Sigma^\infty X_+, \mathbf{H}\,\underline{\mathrm{K}}_*^{\mathrm{MW}}(n)[2n]] \simeq \widetilde{\mathrm{CH}}^n(X).$$

The latter can be computed using the Rost-Schmid complex (cohomological form):

$$\oplus_{y \in X^{n-1}} \mathrm{K}_1^{\mathrm{MW}}(\kappa_y, \nu_y) \xrightarrow{d^{n+1}} \oplus_{x \in X^{(n)}} \mathrm{GW}(\kappa_x, \nu_x) \xrightarrow{d^n} \oplus_{s \in X^{(n-1)}} \mathrm{W}(\kappa_s, \nu_s)$$

[41]This follows from Borel's computation of the K-theory of fields. See [Dég24].

[42]This follows from the conjunction of the fact that Witt groups of a positive characteristic field is 2-torsion, and from the Beilinson conjecture on algebraic K-theory, asserting that the symbol map is rationally an isomorphism for postive characteristic field.

where $X^{(n)}$ is the set of codimension n points of x, and for such a point, $\nu_x = \Lambda^n N_x$ where N_x is the normal bundle of x in $X_{(x)}$. This is given by the *quadratic cycles* on X in codimension n, that is the kernel of d^n, modulo the image of d^{n+1} which one can call the *quadratic cycles rationally equivalent to* 0.

4.1.b. **Milnor-Witt motivic cohomology and higher Grothendieck-Witt groups.**

4.1.3. *Milnor-Witt motivic cohomology.* The Chow-Witt groups play a role similar to that of Chow groups. Chow groups have been extended to higher Chow groups with the aim of having a localization long exact sequence, and the latter are also called motivic cohomology represented by the ring spectrum $\mathbf{H}_{\mathrm{M}}\mathbb{Z}_S$.

Similarly, Chow-Witt groups can be extended to the so-called Milnor-Witt motivic cohomology $\mathbf{H}_{\mathrm{MW}}\mathbb{Z}_k$, which has been defined in [BCD+ar] over a perfect field k, and by pullback over any scheme over such a field. With rational coefficients, we can define the Milnor-Witt motivic cohomology spectrum is nothing else than the rationalization of the sphere spectrum (see [DFJK21]). Thus we will put:

$$\mathbf{H}_{\mathrm{MW}}\mathbb{Q}_S = \mathbb{1}_S \otimes \mathbb{Q}.$$

One has in particular:

$$\mathbf{H}^{2n,n}_{\mathrm{MW}}(X, R) \simeq \widetilde{\mathrm{CH}}^n(X) \otimes_{\mathbb{Z}} R$$

for X smooth over k and $R = \mathbb{Z}$ or S regular and $R = \mathbb{Q}$. Thus in any case, the preceding discussion implies that $\mathbf{H}_{\mathrm{MW}}R_S$ is not orientable.

4.1.4. *Higher Grothendieck-Witt groups.* The last player today will be the higher Grothendieck-Witt ring spectrum GW_S. It was defined by Panin-Walter [PW18] and Schlichting-Tripati [ST15] when 2 is invertible on S, and which has recently been defined in the general case by Calmès, Harpaz, Nardin [CHN24].

Whereas $\mathbf{KGL}_S$ is $(2,1)$-periodic, GW_S is $(8,4)$-periodic. Here is a computation of these groups, for S regular:

$$\mathbf{GW}^{n,i}(S) = GW^i_{2i-n}(S) = \begin{cases} KO_{2i-n} & i \cong 0 \mod 4, \\ KSp_{2i-n} & i \cong 2 \mod 4, \end{cases} \tag{4.15}$$

where KO_* (resp. $\mathbf{KSp}_*$) is the higher hermitian K-theory of symmetric (resp. symplectic) vector bundles: vector bundles equipped with a symmetric (resp. anti-symetric) non-degenerate bilinear form.

One has an exact sequence of motivic spectra:

$$\mathrm{GW}_S(1)[1] \xrightarrow{\gamma_\eta} \mathrm{GW}_S \xrightarrow{f} \mathbf{KGL}_S \xrightarrow{h\circ\gamma_{\beta'}} \mathrm{GW}_S(1)[2]$$

where η is the algebraic Hopf map, f the forgetful map, $h : \mathbf{KGL}_S \to \mathrm{GW}_S$ is the "hyperbolization" map and $\gamma_{\beta'}$ is the multiplication by the inverse of the Bott element β on $\mathbf{KGL}_S$. This implies that η is non zero on hermitian K-theory, and therefore GW_S is not orientable.

4.2. Symplectic orientation after Panin and Walter

4.2.1. *Torsors and algebraic group schemes.* We have seen in the preceding lecture that an orientation on a ring spectrum can be exactly encoded into a ring map from the algebraic cobordism spectrum $\mathbf{MGL}_S$ (see Theorem 2.1.8).

The idea of Panin and Walter, in particular inspired by Balmer's theory of higher Witt groups, is that one can find the appropriate theory of characteristic classes by only asking for Thom classes with respect to vector bundles with an appropriate additional structure. The main players here, used to define hermitian K-theory, are the symmetric and symplectic bundles.[43]

In the talk of Philippe Gille, we learn how to model these extra-structures as torsors on appropriate algebraic linear groups. Here is a picture:

$$\begin{array}{ccccc} \mathrm{SO}_\infty & \longrightarrow & \mathrm{O}_\infty & & \\ & \searrow & & \searrow & \\ & & \mathrm{SL}_\infty & \longrightarrow & \mathrm{GL}_\infty \\ & \nearrow & & & \\ \mathrm{Sp}_\infty & & & & \end{array}$$

The dictionnary is as follows, having fixed a scheme X and working relative to X:

(1) torsors on SL_∞ corresponds to vector bundles V with a trivialization of its determinant $\det(V) = \Lambda^n V$, $n = \mathrm{rk}(V)$.
(2) torsors on Sp_∞ corresponds to vector bundles V with a symplectic form: an anti-symmetric non-degenerate bilinear form $\psi : V \otimes V \to V$. Note that the existence of such a form implies that V has *even rank*.
(3) torsors on O_∞ (resp. SO_∞) corresponds to vector bundles V with a symmetric form: a symmetric non-degenerate bilinear form $\phi : V \otimes V \to V$ (resp. and a trivialization of the deteremінant of V).

Beware that one should be extra careful here: for the first two points, the notion of torsors will not change when we vary the topology from

[43] In the modern theory of hermitian K-theory, both situations are encoded by considering an appropriate biduality structure. In the recent treatment of [CDH+23], this is described by a Poincaré structure on an ∞-category.

Zariski, Nisnevich, étale and fppf. On the contrary, this is false in the third case: there exists étale-locally trivial torsors that are not Zariski-locally trivial.[44]

Thus in the third case, one has to be careful when defining the classifyng space of G, as the difference between Nisnevich and étale matters. It is possible to define Nisnevich-local and étale-local classifying spaces for these groups, we will restrict our attention on the first two cases.

4.2.2. Let us consider the ind-group schemes (over some base S) $G = \mathrm{SL}_\infty, \mathrm{Sp}_\infty$. We can now use the procedure of 2.1.5, using the classifying spaces BSL_n and BSp_{2n} respectively, and their tautological bundles, in place of BGL_n. This defines motivic ring spectra over S:

$$\mathbf{MSL}_S = \mathrm{hocolim}_{n\geq 0} \mathrm{Th}(\gamma_n^{\mathrm{SL}})(-n)[-2n],$$
$$\mathbf{MSp}_S = \mathrm{hocolim}_{n\geq 0} \mathrm{Th}(\gamma_{2n}^{\mathrm{SL}})(-2n)[-4n].$$

Note that according to the above diagram of algebraic groups, one gets morphisms of ring spectra:

$$\mathbf{MSp}_S \to \mathbf{MSL}_S \to \mathbf{MGL}_S.$$

Starting from this definition, we can safely define an G-orientation as an $\mathrm{M}G_S$-algebra structure. With this terminology, a GL-orientaion is nothing else than an orientation as defined in Definition 1.2.1.[45]

With this definition, we get a hierarchy: an Sp-orientation canonically induces an SL-orientation, which in turn canonically induces a GL-orientation.

Both SL-orientations and Sp-orientations are useful in motivic homotopy. On the one hand, SL-orientations are very similar to GL-orientations and, up to considering twists by line bundles, one can define Thom classes for arbitrary vector bundles. They also contain the theory of Euler classes that has been used by Morel and Asok-Fasel to obtain many advances on the *Murthy's conjectures* (see [AF23]).

On the other hand, Sp-orientations have a richer theory of characteristic classes, which is much closer to that of GL-orientations. This allows to transport most of the techniques used in the latter. Therfore, the remaining of this course will focus on the latter.

[44] These two facts boils down to the fact that GL_n, SL_n and Sp_{2n} are *special* in the sense of [Ser54] while SO_n and O_n are not. This was originally proved by Grothendieck (when X is defined over an algebraically closed field): see [Gro58]. For an arbitrary affine smooth group scheme G over some absolute base, this reduces to the question whether $H^1_{\text{ét}}(S, G)$ is trivial or not when S if a local (henselian) scheme. The Grothendieck-Serre conjecture eventually reduces this question to the case where S is the spectrum of a field.

[45] This is a small conflict of terminology, not really problematic anyway.

4.2.a. **Definition.** As said just above, Sp-orientations shared the good properties of GL-orientations. In particular, we let $\mathrm{H}\mathbb{P}^\infty_S = \mathrm{BSp}_2$, the hermitian infinite projective space, classifying the rank two symplectic bundle. It is notable that one can compute the first layer of this ind-scheme, in the pointed $\mathbb{A}^1$-homotopy category:

$$\mathrm{H}\mathbb{P}^1_S \simeq (\mathbb{P}^1_S)^{\wedge,2}.$$

The following theorem, due to Panin and Walter (see [PW23, Th. 1.1]), is an analogue of the results we obtained in the GL-oriented case (Theorem 2.1.8).

Theorem 4.2.3 (Panin-Walter). *Let $\mathbb{E}$ be a ring spectrum over S. Then the following data are in bijective correspondance:*

1. *the cohomology classes $b \in \tilde{\mathbb{E}}^{4,2}(\mathrm{H}\mathbb{P}^\infty_S)$ such that $b|_{\mathrm{H}\mathbb{P}^1_S} = \Sigma^2_T 1_{\mathbb{E}}$.*
2. *The* **MSp**-*algebra structure on $\mathbb{E}$ over S: $\varphi : \mathbf{MSp}_S \to \mathbb{E}$.*
3. *The collection of Thom classes $\mathrm{th}(V,\psi) \in \mathbb{E}^{4n,2n}(\mathrm{Th}(V))$ for symplectic vector bundles (V,ψ) over smooth S-schemes satisfying suitable properties: compatibility with pullbacks, multiplicativity with respect to direct sums, and normalisation.*

In particular, we can declare that an Sp-orientation of $\mathbb{E}$ is an $\mathbb{E}$-cohomology class b as in the above point (1).

Example 4.2.4.
1. Both the Milnor-Witt motivic cohomology spectrum, and the the higher Grothendieck-Witt groups are Sp-orientable (see e.g. [DF21]).
2. Morevoer, it follows from [DFJK21], that any rational motivic ring spectrum is Sp-orientable. Indeed, such an $\mathbb{E}$ is automatically an $\mathbf{H}_{\mathrm{MW}}\mathbb{Q}_S$-module.

4.2.b. **Borel classes.**

4.2.5. *Borel classes.* As in the GL-orientation case, it induces, for any smooth S-scheme X, a map:

$$b_1 : \mathrm{Pic}^{\mathrm{Sp}}(X) := H^1(X, \mathrm{Sp}_2) \to [X_+, \mathrm{BSp}_2]^{un} \to \mathbb{E}^{4,2}(X).$$

which we will call the first Borel classes. Mind that there is no question of being a homomorphism for b_1 here: the left hand-side does not have a group structure as the tensor product of two rank 2 symplectic bundle is not a symplectic bundle!

Nethertheless, a key point of the theory is the following analogue of the projective bundle theorem. Consider a sumplectic bundle (V,ψ) over a smooth S-scheme X. We can define the associated projective sympelctic bundle $\mathrm{H}\mathbb{P}(V,\psi)$ as the open subscheme of the Grassmannian scheme $\mathrm{Gr}(2,V)$ on which the restriction of ψ to the canonical sub-bundle of rank 2 is non-degenerate.

We let U be the tautological rank 2 bundle on $\mathrm{H}\mathbb{P}(V,\psi)$. By definition, it is equipped with a symplectic structure ψ_U coming from the restriction of ψ.

Theorem 4.2.6 (Panin-Walter)**.** *Consider the above notation and assume that V has rank $2n$. Let $p : \mathrm{H}\mathbb{P}(V,\psi) \to X$ be the canonical projection, $(\mathbb{E}, b)$ be an* Sp*-oriented ring spectrum and $b = b_1(U,\psi)$ be the associated first Borel class.*

Then the following map is an isomorphism of bi-graded $\mathbb{E}^{**}(X)$-modules:

$$\bigoplus_{i=0}^{n} \mathbb{E}^{**}(X) \to \mathbb{E}^{**}\big(\mathrm{H}\mathbb{P}(V,\psi)\big), x_i \mapsto p^*(x_i).b^i.$$

See [PW21, Th. 8.2] for a proof. As a consequence, we can define the following higher Borel classes $b_i(V,\psi) \in \mathbb{E}^{4i,2i}(X)$

$$\sum_{i=0}^{n}(-1)^i.b_i(V,\psi).b^{n-i} = 0, b_0(V,\psi) = 1, \forall i > n, b_i(V,\psi) = 0. \tag{4.16}$$

These classes satisfy the exact analog properties of their parent, the Chern classes. Moreover, one deduces from the above theorem a symplectic splitting principle (analog to Proposition 1.2.8), which gives an important tool for computations. We refer the reader to [DF21, §2.2] for details. But note finally that it will be usefull to introduce the total Borel class:

$$b_t(V,\psi) = 1 + b_1(V,\psi).t + \dots$$

as an element of $\mathbb{E}^{**}(X)[t]$. If we give the formal variable t degree $(-4,-2)$, this class is homogeneous of bidegree $(0,0)$.

Example 4.2.7. The motivic ring spectrum $\mathbf{H}_\sigma$ representing the singular cohomology of the real points, attached to $\sigma : k \to \mathbb{R}$ and defined in Example 1.1.6(3), is Sp-orientable. This follows from the existence of a (quadratic) cycle class map from Chow-Witt groups as was constructed in [HWXZ21].

Given a vector bundle V, one defines its symplectification as:

$$\mathcal{H}(V) := \left(V \oplus V^\vee, \begin{pmatrix} 0 & 1 \\ -\mathrm{can} & 0 \end{pmatrix}\right).$$

One then defines the *Pontryagin classes* of V as:

$$p_i(V) = b_i\big(\mathcal{H}(V)\big).$$

It has not been checked, but it seems likely that the Pontryagin classes defined above for the Sp-oriented ring spetrum $\mathbf{H}_\sigma$ does coincide with the classically defined Pontryagin classes.

4.2.c. **Formal ternary laws.**

4.2.8. Let us write generically $\mathfrak{V} = (V, \psi)$ the symplectic bundles that will appear subsequently.

As already mentioned, the tensor product $\mathfrak{V}_1 \otimes \mathfrak{V}_2$ of two symplectic bundle, is not a symplectic bundle, but a symmetric one. A priori, that forbids transporting the theory of FGL into the symplectic case. On the other hand, a triple tensor product of symplectic bundles is symplectic. This motivates Walter to introduce *formal ternary laws*, abbreviated to FTL, in this context.

But mind that a triple product $\mathfrak{V}_1 \otimes \mathfrak{V}_2 \otimes \mathfrak{V}_3$ has rank 8. Consequently, it has a priori four non-trivial Borel classes. What we concretely get out of these considerations is the following construction, for which we refer to [DF21, §2.3].

Proposition 4.2.9. *Let $(\mathbb{E}, b)$ be an* Sp*-oriented motivic ring spectrum over S. Put $\mathbb{E}^{**} = \mathbb{E}^{**}(S)$.*

There exists a a power series $F_t^b(x, y, z)$ in $\mathbb{E}^{**}[[x, y, z]][t]$ of the form:

$$F_t^b(x, y, z) = 1 + \sum_{i=1}^{4} F_i(x, y, z).t^i$$

such that for any triple of symplectic bundles $(\mathfrak{V}_1, \mathfrak{V}_2, \mathfrak{V}_3)$, the following relation holds:

$$b_t(\mathfrak{V}_1 \otimes \mathfrak{V}_2 \otimes \mathfrak{V}_3) = F_t^b\big(b_1(\mathfrak{V}_1), b_1(\mathfrak{V}_2), b_1(\mathfrak{V}_3)\big).$$

Proof. The method of proof of this proposition, goes as in the GL-oriented case: we use the symplectic projective bundle theorem and the canonical map:

$$\mathrm{H}\mathbb{P}^\infty_S \times_S \mathrm{H}\mathbb{P}^\infty_S \times_S \mathrm{H}\mathbb{P}^\infty_S \to \mathrm{H}\mathbb{P}^\infty_S$$

which is a structure of abelian ternary group on the ind-scheme $\mathrm{H}\mathbb{P}^\infty_S$. The formula stated in the proposition then follows by the fact $\mathrm{H}\mathbb{P}^\infty_S = \mathrm{BSp}_2$ classifies the symplectic bundles of rank 2. □

4.2.10. *Multi-valued series.* The power series $F_t(x, y, z)$ is made of 4-components and has 3 variables. We call this a $(4, 3)$-series, inspired by Buchstaber's theory of 2-valued formal group laws which are actually $(2, 2)$-series.[46] A delicate algebraic point in the theory of (n, m)-series

[46]In this point of view, formal group laws are $(1, 2)$-series!

(also called "multi-valued series") is the description of a substitution procedure, based on an algebraic splitting principle.[47] If one admits this construction, here are the properties of the $(4,3)$-series $F_t(x,y,z)$ arising above.

A final observation is that all abelian groups that arise from morphisms in motivic stable homotopy are in fact algebras over the ring:

$$\mathbb{Z}_\epsilon = \mathbb{Z}[\epsilon]/(\epsilon^2 - 1).$$

Indeed, ϵ in the above ring corresponds to the element ϵ described in 1.2.13.[48] Therefore, it is natural to adopt the following definition ([CDFH22, Def. 3.1.5], [DF21, Def. 3.1.2]).

Definition 4.2.11. Let R be a $\mathbb{Z}_\epsilon$-algebra. A *formal ternary law* with coefficients in R is $(4,3)$-series with coefficients in R

$$F_t(x,y,z) = 1 + F_1(x,y,z)t + F_2(x,y,z)t^2 + F_3(x,y,z)t^3 + F_4(x,y,z)t^4$$

satisfying the following properties:

(1) *Neutral element.* $F_t(x,0,0) = (1+xt)^2(1-\epsilon xt)^2$.
(2) *Semi-neutral element.* $F_4(x,x,0) = 0$.
(3) *Symmetry.* The element $F_t(x,y,z)$ of $R[[x,y,z]][t]$ is fixed under the obvious action of the symmetric group $\mathfrak{S}(x,y,z)$.
(4) *Associativity.* Given formal variables (x,y,z,u,v), one has the following equality of $(16,5)$-series:

$$F_t\big(F_t(x,y,z),u,v\big) = F_t\big(x,F_t(y,z,u),v\big)$$

using the substitution operations of multi-valued series.
(5) *ϵ-Linearity.* $F_t(-\epsilon x,y,z) = F_{-\epsilon t}(x,y,z)$.

We frequently display an FTL by its coefficients:

$$F_t(x,y,z) = 1 + \sum_{i,j,k\geq 0, 1\leq l\leq 4} a^l_{ijk} x^i y^j z^k t^l. \tag{4.17}$$

We define the *degree* of $F_t(x,y,z)$ as the integer:

$$d = \max\{i+j+k-l \mid a^l_{ijk} \neq 0\}.$$

This definition is made so that the $(4,3)$-series $F^b_c(x,y,z)$ appearing in the preceding proposition is an FTL with coefficients in the $\mathbb{Z}_\epsilon$-algebra $\mathbb{E}^{**}(S)$. It is called the FTL associated with $(\mathbb{E},b)$.

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[47]To explain it, recall that the symplectic splitting principle tells that one can formally associates to some total Borel class $b_t(\mathfrak{V})$ some Borel roots β_i such that: $b_c(\mathfrak{V}) = \prod_i(1+\beta_i.t)$. One can then use the algebra of symmetric functions to relate the b_i with the β_i. See [CDFH22, §2.1] for details.

[48]Observe also that $\mathrm{GW}(\mathbb{Z}) \simeq \mathbb{Z}_\epsilon$!

Example 4.2.12. The following FTL are examples with bounded degree. As such, they are respective analogues of the additive and multiplicative FGL.

(1) We have an FTL of *degree* 0 with coefficients in $\mathbb{Z}_\epsilon$ whose non-zero coefficients are:

$$\begin{array}{llll} a^1_{100} = 2(1-\epsilon) & & & \\ a^2_{200} = 2(1-2\epsilon) & a^2_{110} = 2(1-\epsilon) & & \\ a^3_{300} = 2(1-\epsilon) & a^3_{210} = -2(1-\epsilon) & a^3_{111} = 8(2-3\epsilon) & \\ a^4_{400} = 1 & a^4_{310} = -2(1-\epsilon) & a^4_{220} = 2(1-2\epsilon) & a^4_{211} = 2(1-\epsilon). \end{array}$$

This is expected to be the only (up to the automorphism $\epsilon \mapsto -\epsilon$) FTL of degree 0. We proved this after inverting 2 in see [CDFH22, Th. 3.1.12]. By analogy, this FTL is called the *additive FTL*.

We proved in [DF21] that this is the FTL associated with Chow-Witt groups, *i.e.* the motivic ring spectrum $\mathbf{H}_{\mathrm{MW}}\mathbb{Z}_k$ with its canonical Sp-orientations, at least when $2, 3$ are invertible in the base field k. With rational coefficients, this restriction can be lifted: for any S, the Sp-oriented spectrum $\mathbf{H}_{\mathrm{MW}}\mathbb{Q}_S = \mathbb{1}_S \otimes \mathbb{Q}$ has the additive formal ternary laws.

(2) We have an FTL of *degree* 2, having parameters τ and γ, *i.e.* with coefficients in the polynomial ring

$$\mathbb{Z}_\epsilon^{mul} := \mathbb{Z}_\epsilon[\tau, \gamma^{\pm 1}]/\langle \tau^2 - 2(1-\epsilon)\gamma, (1+\epsilon)\tau \rangle,$$

whose non-zero coefficients are, in addition to the ones appearing in the preceding law:

$$\begin{array}{llll} a^1_{110} = \tau\gamma^{-1} & a^1_{111} = \gamma^{-1} & & \\ a^2_{210} = 2\tau\gamma^{-1} & a^2_{111} = -3\tau\gamma^{-1} & a^2_{220} = \gamma^{-1} & \\ a^3_{310} = \tau\gamma^{-1} & a^2_{220} = -2\tau\gamma^{-1} & a^3_{211} = 3\tau\gamma^{-1} & a^3_{220} = \gamma^{-1} \\ a^4_{311} = -\tau\gamma^{-1} & a^4_{221} = 2\tau\gamma^{-1} & a^4_{222} = \gamma^{-1}. & \end{array}$$

It was proved in [FH23, Theorem 6.6] that this is the FTL associated with hermitian K-theory GW_k. For this reason, we call it the *multiplicative FTL*.

4.2.13. *The Walter ring.* The theory of FTL is very involved, though analogous to that of FGL from what we have sen so far. It is possible to define (stric) isomorphisms of FTL, and to get a category $\mathcal{FTL}$. For example, there is a canonical functor:

$$\mathcal{FGL} \to \mathcal{FTL}$$

which reflects the fact that GL-oriented implies Sp-oriented. One also gets in the middle of this functor the category of Buchstaber's 2-valued formal group laws (of a certain form). We refer the reader to [CDFH22].

Let us mention one last fact. The category of FTL admits an initial object, $(\mathcal{W}, \mathbb{F}_t^{\mathcal{W}}(x,y,z))$ and we call the $\mathbb{Z}_\epsilon$-algebra $\mathcal{W}$ the *Walter ring*. At this point, we do not know much about the Walter ring. We have tried to make explicit computations of this ring, but even after bounding the coefficients, the complexity raise very quickly above the reasonable (see the appendix of [CDFH22] for the algorithm).

Nevertheless, it is interesting to point out that the theory gives us two morphisms:

$$\mathcal{W} \to \mathbf{MSp}^{2*,*}(\operatorname{Spec}\mathbb{C}) \to \pi_{-2*}(MSp)$$
$$\mathcal{W} \to \mathbf{MSp}^{2*,*}(\operatorname{Spec}\mathbb{R}) \to \pi_{-*}(MU)$$

using complex and real realizations respectively, and results of [BH21]. One can wonder how the generators of the Walter ring are related to the rather mysterious ring of symplectic cobordism $\pi_{-2*}(MSp)$ through the first map.

Similarly, one could expect, by analogy with the GL-oriented case, that for any field k, the canonical map:

$$\mathcal{W} \to \mathbf{MSp}^{2*,*}(\operatorname{Spec} k)$$

is an isomorphism.

4.3. A quadratic Hirzebruch-Riemann-Roch formula

In this last section, we will explain how to get a quadratic version of the classical Hirzebruch-Riemann-Roch theorem, which allows to compute Euler characteristic of vector bundles in terms of the degree of some map. In fact, recall that the (qsingular) HRR formula is the particular case of the GRR formula obtained by considering the Gysin map with respect to a projective lci variety $p : X \to \operatorname{Spec}(k)$ over some field k (or Dedekind scheme). We will have to introduce three players: the Borel character, the associated Todd class, and the Euler characteristic of symmetric/symplectic bundles.

Underlying this particular quadratic formula, there is a general theory of symplectically oriented fundamental classes, isomorphisms of FTL associated with morphisms of ring spectra (or simply change Sp-orientations). And there is a general Grothendieck-Riemann-Roch formula in the style of **??**. We refer the reader to [DF21] for such a formula.

4.3.a. **The Borel character.**

4.3.1. It was mentioned earlier that Chow-witt groups and hermitian K-theory bear a similar relation than Chow groups and K-theory. Indeed, by studying some operations on hermitian K-theory, we were able with Jean Fasel to build an isomorphism of (rational) ring spectra which is a "quadratic" analog of the Chern character (Example 3.2.4).

Theorem 4.3.2. *For any scheme S, there exists an isomorphism of motivic ring spectra of the form:*

$$\mathrm{bo}_t : \mathrm{GW}_S \to \bigoplus_{n\in\mathbb{Z}} \mathbf{H}^{(n)}_{\mathrm{MW}}\mathbb{Q}_S(2n)[4n]$$

where we put:

$$\mathbf{H}^{(n)}_{\mathrm{MW}}\mathbb{Q}_S = \begin{cases} \mathbf{H}_{\mathrm{MW}}\mathbb{Q}_S & n \text{ odd,} \\ \mathbf{H}_{\mathrm{M}}\mathbb{Q}_S & n \text{ even.} \end{cases}$$

Moreover, the following diagram commutes:

$$\begin{array}{ccc} \mathrm{GW}^{\mathbb{Q}}_S & \xrightarrow{\mathrm{bo}_t} & \bigoplus_{n\in\mathbb{Z}} \mathbf{H}^{(n)}_{\mathrm{MW}}\mathbb{Q}_S(2n)[4n] \\ {\scriptstyle f}\downarrow & & \downarrow \\ \mathbf{KGL}^{\mathbb{Q}}_S & \xrightarrow{\mathrm{ch}_t} & \bigoplus_{m\in\mathbb{Z}} \mathbf{H}_{\mathrm{M}}\mathbb{Q}_S(m)[2m]. \end{array}$$

where the left vertical map is the forgetful map, and the right one maps $\mathbf{H}^{(n)}_{\mathrm{MW}}\mathbb{Q}_S(2n)[4n]$ to the $m = 2n$ factor $\mathbf{H}_{\mathrm{M}}\mathbb{Q}_S(2n)[4n]$, by either the modulo η-map if n is even, or the identity if n is odd.

Concretely, the Borel character on the part $\mathrm{GW}^{4,2}(X) = \mathbf{KSp}_0(X)$ of the higher Grothendieck-Witt group associates to a symplectic bundle $\mathfrak{V} = (V, \psi)$ an element of the following form:

$$\mathrm{bo}_t(V,\psi) = 2d + \tilde{\chi}_2(V,\psi) + \frac{1}{4!}\chi_4(V) + \frac{1}{\psi_6!}\tilde{\chi}_6(V,\psi) + \dots$$

in

$$\mathrm{CH}^0(X)_{\mathbb{Q}} \oplus \widetilde{\mathrm{CH}}^2(X)_{\mathbb{Q}} \oplus \mathrm{CH}^4(X)_{\mathbb{Q}} \oplus \widetilde{\mathrm{CH}}^6(X)_{\mathbb{Q}} \oplus \dots$$

Here, the element $\psi_{2+2n}!$ is an explicit quadratic form, whose rank is $(2+2n)!$

Remark 4.3.3. If we start from $\mathrm{GW}^{0,0}(X) = \mathrm{KO}_0(X)$, then one obtains a similar form, but one has to start from $\widetilde{\mathrm{CH}}^0(X) \simeq \mathrm{GW}(\kappa(X))$ if X smooth connected, and alternate Chow groups with Chow-Witt groups.

4.3.b. **Todd classes.** The Borel character bo_t is a morphism of ring spectra, from the Sp-oriented ring spectrum GW which has the multiplicative FTL to an Sp-oriented ring spectrum which has the additive FTL. As in the case of the Chern character, it corresdponds to a strict isomorphims between those two FTL. At the moment, we do not know that such an isomorphism is unique, nor it is a kind of exponential isomorphism. However, we can define an associated todd class and compute it as follows.

Proposition 4.3.4. *There exists a Todd class morphism:*

$$\widetilde{\mathrm{td}} : \mathbf{KSp}_0(X) \to \widetilde{\mathrm{CH}}^0(X)_{\mathbb{Q}} \oplus \mathrm{CH}^2(X)_{\mathbb{Q}} \oplus \widetilde{\mathrm{CH}}^4(X)_{\mathbb{Q}} \oplus \mathrm{CH}^6(X)_{\mathbb{Q}} \oplus \dots$$

which sends $+$ to $\times$ and which, for any symplectic bundle (V,ψ) of rank $2n$, satisfies the relation

$$\mathrm{ch}_t\big(b_n(V,\psi)\big) = \widetilde{\mathrm{td}}(V,\psi).b_n(V,\psi).$$

Moreover, in even degrees and projected on the minus part (=Witt part), this element satisfies the relation:

$$\widetilde{\mathrm{td}}_-(V,\psi) = \frac{t/2}{\sin(t/2)}(t = e(V,\psi))$$

where $e(V,\psi)$ is the Euler class in the Witt part $\widetilde{\mathrm{CH}}^{2n}(X)_{\mathbb{Q}-}$.

See [DF21, Prop. 4.2.3].

4.3.c. **Euler characteristic of symplectic bundles.**

4.3.5. Let $p : X \to \mathrm{Spec}(k)$ be a projective lci morphism of dimension $d = 2n$, where k is a local regular ring. We assume that the virtual tangent bundle τ_X admits a symplectic orientation[49]: there exists a class $\tilde{\tau}_X \in \mathbf{KSp}_0(X)$ and an isomorphism $f(\tilde{\tau}_X) \simeq \tau_X$, where $f : \mathbf{KSp}_0(X) \to K_0(X)$ is the forgetful map.

Let $\omega_X = \det(\tau_X)$ be the canonical bundle of X. We recall that this symplectic orientation actually induces an orientation in the classical sense of p: that is, an isomorphism: $\epsilon_X : L^{\otimes 2} \to \omega_X$.

The last element we need is a computation of

we

Definition 4.3.6. Consider the above notation as well as the Gysin map:

$$p_* : \mathrm{GW}^{4m,2m}(X) \to \mathrm{GW}^{4(m-n),2(m-n)}(k)$$

associated with the symplectic orientation of X/k.

[49]It seems clear that we in fact need only an orientation in the sense of Barge-Morel to get the next formula, as well as the quadratic HRR formula, work.

Given an element $v = [V, \varphi]$ of the left hand-side, which is either a symetric (n even) or a symplectic (n odd) bundle, we define the quadratic Euler characteristic of v as the element:

$$\tilde{\chi}(X, \tilde{\tau}_X; V, \varphi) = p_*([V, \varphi]) \in \mathrm{GW}^{4(m-n),2(m-n)}(k).$$

Note this is 0 if $m-n$ is odd, and the class of a symmetric form over k if $m-n$ is even.

4.3.7. We consider the notation of the preceding definition. Here is a way to compute this Euler characteristic. First, the orientation ϵ_X associated with $\tilde{\tau}_X$ induces a Thom isomorphism:

$$\begin{aligned} \mathrm{GW}^{4m,2m}(X) \simeq \mathrm{GW}^{4m,2m}(X, L^{\otimes 2}) &\xrightarrow{\epsilon_X} \mathrm{GW}^{4m,2m}(X, \omega_X) \\ [V, \varphi] &\mapsto [V \otimes L, \varphi'_L] \end{aligned}$$

where we have define the following form:

$$\varphi_L : V \otimes V \otimes L \otimes L \xrightarrow{\varphi \otimes \mathrm{Id}_{L \otimes L}} L^{\otimes 2} \xrightarrow{\epsilon_X} \omega_X.$$

By transposition, one deduces an isomorphism:

$$\varphi'_L : V \otimes L \to (V \otimes L)^\vee \otimes \omega_X.$$

One deduces a form on cohomology as follows:

$$\varphi'_{L,\epsilon} : H^n(X, V \otimes L) \xrightarrow{\varphi'_{L*}} H^n(X, (V \otimes L)^\vee \otimes \omega_X) \simeq H^n(X, V \otimes L)^*$$

where the last isomorphism follows by Grothendieck duality.

Proposition 4.3.8. *Using the above notation, one gets:*

$$\begin{aligned} \tilde{\chi}(X, \tilde{\tau}_X; V, \varphi) = &(-1)^{m+n}[H^n(X, V \otimes L), \varphi'_{L,\epsilon}] \\ &+ \left(\sum_{i=0}^{n-1} (-1)^{(m+i)} \dim_k H^i(X, V \otimes L) \right) \cdot h \end{aligned}$$

in $\mathrm{GW}^{4(m-n),2(m-n)}(k)$.

4.3.d. **Main theorem.** We can now formulate the quadratic HRR theorem, which follows from the general quadratic GRR theorem associated with the Borel character and the relevant Sp-orientations.

Theorem 4.3.9. *Let X/k be an lci projective variety of even dimension $d = 2n$ over a local regular ring k, such that X is regular. We assume that the virtual tangent bundle τ_X of X/k admits a stable* Sp*-orientation $\tilde{\tau}_X$. We let ϵ_X be the orientation of X/k associated with $\tilde{\tau}_X$.*

Let (V, φ) be either a symplectic bundle if $d = 2 \pmod 4$ or a symmetric bundle if $d = 0 \pmod 4$. Then the following equality holds in $\mathrm{GW}(k)$

$$\tilde{\chi}(X, \tilde{\tau}_X; V, \varphi) = \widetilde{\deg}_{\tilde{\tau}_X} \left(\mathrm{td}_{\mathrm{bo}_t}(\tilde{\tau}_X) . \mathrm{bo}_t(V, \varphi)\right)$$

where $\widetilde{\deg}_{\tilde{\tau}_X} \colon \widetilde{\mathrm{CH}}^d(X) \to \mathrm{GW}(k)$ *is the (quadratic) degree map (push-forward) associated with the symplectic orientation $\tilde{\tau}_X$ (equivalently the orientation ϵ_X) of X/k.*

In the case where k is a field of characteristic not 2, the above formula, as well as the other formulations of the general quadratic GRR theorems proved in [DF21], should be compared with the formulas obtained by Marc Levine and Apron Raskit in [LR20, Th. 1.3, 8.6, 8.7].

Example 4.3.10. Suppose that $d = 2$ and that X/k is a K3 surface. We consider the symplectic orientation of X/k obtained by choosing an isomorphism $\tilde{\tau}_X : \mathcal{O}_X \to \omega_{X/k}$. If (V, ψ) is a symplectic bundle of rank $2r$ over X, we finally obtain

$$\begin{aligned} \tilde{\chi}(X, \tilde{\tau}_X; V, \psi) &= \widetilde{\deg}_{\tilde{\tau}_X} \left((2r + e(V, \psi)) \cdot (1 + \frac{c_2(T_{X/k})}{24} h) \right) \\ &= \widetilde{\deg}_{\tilde{\tau}_X} \left(r \cdot \frac{c_2(T_{X/k})}{12} h + e(V, \psi) \right) \\ &= \deg \left(r \cdot \frac{c_2(T_{X/k})}{12} \right) \cdot h + \widetilde{\deg}_{\tilde{\tau}_X}(e(V, \psi)). \end{aligned}$$

Using the well-known fact that $\deg(c_2(\Omega_{X/k})) = 24$, we finally obtain the formula in $\mathrm{GW}(k)$:

$$\tilde{\chi}(X, \tilde{\tau}_X; V, \psi) = 2r \cdot h + \widetilde{\deg}_{\tilde{\tau}_X}(e(V, \psi)).$$

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