

Conventions. Schemes will be Noetherian. Smooth will mean smooth of finite type. Unless stated otherwise, the Grothendieck topology on $\mathcal{S}m_S$ is the Nisnevich topology. Therefore, sheaves will mean sheaves for the Nisnevich topology.

We use the language of ∞ -categories.¹ We let Cat_∞ (resp. Cat_∞^\otimes) be the ∞ -category of presentable ∞ -categories with left adjoint ∞ -functors (resp. presentable symmetric closed monoidal ∞ -category with left adjoint and symmetric monoidal ∞ -functors). All our ∞ -categories will be presentable ∞ -categories. Similarly, monoidal ∞ -categories will be presentable monoidal ∞ -categories. On the other hand, we mostly work in the associated homotopy category in this course.

Monoidal means symmetric monoidal. All our monoids are commutative. We denote by $\mathbb{1}_S$ the “sphere spectrum” over S *i.e.* the unit of the monoidal structure on $\mathrm{SH}(S)$.

Unless stated otherwise, spectrum means motivic spectrum, and ring spectrum means commutative motivic spectrum.

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¹One can obtain an explicit presentation of the categories here by using classical model categories on simplicial sheaves: the injective and Nisnevich-local model category structures on simplicial sheaves, and their Bousfield \mathbb{A}^1 -localization.

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Cours 1. Oriented spectra and Chern classes

INTRODUCTION

The theory of *characteristic classes* of *fiber bundles* arose at exactly the same time than (singular) cohomology, in 1935. That year, Stiefel (in his PhD) and Whitney both introduced the notion of fiber bundle and some associated characteristic class.² Meanwhile, at the Moscow international conference on topology, Alexander and Kolmogorov independently introduced cohomology and the (soon to be called) cup-product.

The history of the subject of characteristic classes was then marked by the introduction of *Pontryagin classes*, out of the computation of the homology of real grassmanianns, by Pontryagin in 1942, and by the introduction of *Chern classes*, obtained through the determination of the cohomology of complex Grassmanians by Chern in 1946. A last event I want to mention is the course "characteristic classes" given at the University of Princeton by John Milnor in 1957.³

Here is a list of the characteristic classes that emerged from the works mentioned above:⁴

name	fiber b.	notation	group
Stiefel-Whitney	smooth real v.b.	w_i	$H^i(B, \mathbb{Z}/2)$
Pontryagin	smooth real v.b.	p_i	$H^{4i}(B, \mathbb{Z})$
Chern	smooth complex v.b.	c_i	$H^{2i}(B, \mathbb{Z})$

²The terminology "fiber bundle" is due to Stiefel, though he actually only considered smooth real vector bundles, while it was Whitney that formally introduced the so-called characteristic classes.

³Notes by Stasheff were available at that time. They were finally published in 1974, [MS74].

⁴Recall that Pontryagin classes are actually particular cases of Chern classes according to the formula: $p_i(V) = (-1)^i \cdot c_{2i}(V \otimes_{\mathbb{R}} \mathbb{C})$, where $V \otimes_{\mathbb{R}} \mathbb{C}$ is the complexification of the real vector space V .

Note that fiber bundle had already appeared: first in works of E. Cartan on Lie groups and their associated homogeneous spaces, and in the work of Stiefel, slightly earlier in 1933, who was interested in constructing new 3-dimensional varieties (in view of the Poincaré conjecture). Recall that in the most general form, a fiber bundle is a map $p : E \rightarrow B$ such that there exists an open cover $W \rightarrow B$ and a W -homeomorphism: $(F \times W) \rightarrow E \times_B W$ for some space F .⁵ One calls B , E , F respectively the base (space), the total space and the fiber (space) of the fiber bundle.⁶

Example 1.0.1. Here are some of the most famous examples of fiber bundles:

- *tangent bundles.* $p : TM \rightarrow M$, projection from the tangent bundle of a smooth (resp. analytic) manifold M . This is a particular case of smooth real (resp. complex analytic) vector bundles.
- *homogeneous spaces.* for G a Lie group, and $H \subset G$ a closed subgroup, $p : G \rightarrow G/H$. This is a particular case of a principal G -bundle.
- *Covering spaces.* $P \rightarrow X$. The fiber is then a *discrete* space.
- The Möbius strip T is (non trivially!) fibered over S^1 , the map $T \rightarrow S^1$ being the projection.
- The Hopf fibration: $S^3 \rightarrow S^2$, with fiber S^1 .

In the previous list, only vector bundles were considered. In topology, more general fiber bundles naturally appear in the so-called *obstruction theory*. They arise as morphisms in the Postnikov tower, in good cases (simple, or more generally nilpotent spaces). The attendees have already seen this theory at work in the talk of Aravind Asok: primary and secondary obstructions can be seen as characteristic classes. In this course however, we will focus on algebraic vector bundles, in order to draw a picture similar to the above table.

Characteristic classes are invariant under isomorphism of fiber bundles. In particular, they can differentiate the homotopy type of the total space. However, they are far from determining this homotopy

⁵Variants arise first by working in other categories than topological spaces. In algebraic geometry, one also considers covers from various (Grothendieck) topologies: Zariski, Nisnevich, étale, fppf (mainly).

⁶Remark that such a fiber space is in particular a Hurewicz fibration, and therefore a Serre fibration. So one sometimes abusively says “fibration” for “fiber bundle”.

type (let alone the diffeomorphism type), even if one adds the homotopy type of the base.⁷ In his groundbreaking 1954 work on cobordism, Thom proved a therefore very surprising fact: the cobordism class of an unoriented closed smooth manifold M is completely determined by the so-called Pontryagin numbers, which are computed through Pontryagin classes of the tangent space of M . This was the beginning of a deep revolution in algebraic topology, which contribute to led to generalized cohomology theories, aka *spectra* such as cobordism, complex (real, Morava,...) K-theory, elliptic cohomology, and the beautiful picture painted by chromatic homotopy theory.

In this talk, we will consider the theory of characteristic classes as it was developed in motivic homotopy theory, after Voevodsky, Morel, Panin, Levine, and many more! The authors interest on the subject arose during his PhD under the supervision of Fabien Morel, during the years 1999-2002. This interest has grown during all my carrier (as can be seen in one's bibliography). I would like to seize this opportunity to thank Fabien again to having shared his visions on, and led me to, this wonderful world of motivic homotopy.

1.1. STABLE MOTIVIC HOMOTOPY

1.1.a. Motivic spectra.

1.1.1. Stable homotopy theory. Recall from the preceding talks that the \mathbb{A}^1 -homotopy category $H^{\mathbb{A}^1}(S)$ over a scheme S is obtained by considering the ∞ -topos $\mathrm{Sh}^\infty(\mathcal{S}m_S)$ of Nisnevich sheaves over the smooth site $\mathcal{S}m_S$ and by localizing it further with respect to \mathbb{A}^1 -homotopy: that is we invert for any smooth S -scheme X the maps $\mathbb{A}_X^1 \rightarrow X$ in $\mathrm{Sh}^\infty(\mathcal{S}m_S)$ via the Yoneda embedding.

On the associated ∞ -category $H_\bullet^{\mathbb{A}^1}(S)$ of pointed objects in $H^{\mathbb{A}^1}(S)$, we even get a symmetric monoidal ∞ -category where the tensor product is the so-called smash product.

We have seen in the talk of F. Morel that there are several models of spheres in motivic homotopy theory: the simplicial sphere S^1 , the multiplicative group $(\mathbb{G}_m, 1)$, and the projective line (\mathbb{P}^1, ∞) . All objects are considered in $H_\bullet^{\mathbb{A}^1}(S)$ without indicating the bases scheme S (which plays no specific role here) in the notation. And they are related by the relation:

$$(1.1) \quad \mathbb{P}^1 \simeq S^1 \wedge \mathbb{G}_m$$

⁷This can already be seen in the classification of Seifert fibrations, mentioned above.

As in classical topology, we obtain the stable motivic homotopy category by \otimes -inverting the third model of sphere, \mathbb{P}^1 . For completeness, we will now state the main theorem that will give us our fundamental category (see also the talk of Kirsten Wickelgren).

Theorem 1.1.2 (Robalo). *Let S be any scheme. There exists a universal presentable monoidal ∞ -category $\mathrm{SH}(S)$ equipped with a monoidal ∞ -functor:*

$$\Sigma^\infty : \mathbf{H}_\bullet^{\mathbb{A}^1}(S) \rightarrow \mathrm{SH}(S)$$

*which admits a right adjoint Ω^∞ and such that $\Sigma^\infty \mathbb{P}^1$ is \otimes -invertible.*⁸

Actually, the proposition could be stated more abstractly for an arbitrary presentable monoidal ∞ -category and an arbitrary object \mathbb{S} . Under, this form the proof is due to Robalo: [Rob15].

1.1.3. It follows from the construction, and the isomorphism (1.1) that, all the possible spheres $\mathbb{S} = \mathbb{P}_S^1, \mathbb{G}_{m,S}, S^1$, becomes \otimes -invertible after applying Σ^∞ .

The resulting ∞ -category acquires a very important property: it is stable in the sense of [Lur17, Def. 1.1.1.9]. This implies that the associated homotopy category admits a triangulated structure (see [Lur17, 1.1.2.13]). Note that the suspension functor for this triangulated structure is given by the formula:

$$\mathbb{E}[1] = \mathbb{E} \otimes \Sigma^\infty S^1.$$

Definition 1.1.4. The monoidal ∞ -category $\mathrm{SH}(S)$ is called the *stable motivic homotopy category* over S . Its objects are called *motivic spectra* over S .

The unit object with respect to the monoidal structure is denoted by $\mathbb{1}_S$. One defines the *Tate twist* as $\mathbb{1}_S(1) = \Sigma^\infty \mathbb{P}^1[-2] = \Sigma^\infty \mathbb{G}_{m,S}[-1]$. By construction, this is a \otimes -invertible objects in $\mathrm{SH}(S)$ so that one also denotes by $?(n)$ the n -th tensor product with respect to this object.

If this does not cause confusion, we will denote by

$$[\mathbb{E}, \mathbb{F}]_S = \mathrm{Hom}_{\mathrm{SH}(S)}(\mathbb{E}, \mathbb{F}) = \pi_0 \mathrm{Map}_{\mathrm{SH}(S)}(\mathbb{E}, \mathbb{F})$$

the abelian group of morphisms in the homotopy category associated to the ∞ -category $\mathrm{SH}(S)$. We usually even drop the index S in the notation.

It might be useful to have in mind the classical model for motivic spectra⁹ given here without taking care about the monoidal structure.

⁸An object X in a monoidal ∞ -category is \otimes -invertible if the ∞ -functor $X \otimes ?$ is an equivalence of categories.

⁹That is the objects of an underlying model category whose associated ∞ -category is equivalent to the above one.

A model for a motivic spectrum \mathbb{E} is the data of a sequence $(\mathbb{E}_n)_{n \geq 0}$ where \mathbb{E}_n is a pointed simplicial Nisnevich sheaf together with suspension maps:

$$\mathbb{P}^1 \wedge \mathbb{E}_n \rightarrow \mathbb{E}_{n+1}.$$

1.1.b. Representable cohomology theories. For us, the main function of the stable homotopy category is that its objects, the \mathbb{P}^1 -spectra, represent cohomology theory. The originality of the theory is that these cohomology theories are *bigraded*.

Definition 1.1.5. Cohomology theories. Let \mathbb{E} be a motivic spectrum over S . For any smooth S -scheme X and any pair of integers $(n, i) \in \mathbb{Z}^2$, one defines the \mathbb{E} -cohomology of X in degree n and twists i as:

$$\mathbb{E}^{n,i}(X) = [\Sigma^\infty X_+, \mathbb{E}(i)[n]].$$

These cohomologies have the distinctive features of being contravariant, additive, \mathbb{A}^1 -homotopy invariant and \mathbb{P}^1 -stable. Moreover, one gets long exact sequences of Mayer-Vietoris type but with respect to Nisnevich distinguished squares.

Example 1.1.6. There are many examples of cohomology theories which are representable in the stable motivic homotopy category, over a given base field $S = \text{Spec}(k)$.

- (1) All the classical Weil cohomologies admits canonical extensions over smooth k -schemes which are representable.¹⁰
 - $\text{char}(k) = 0$: algebraic de Rham cohomology;
 - $\text{char}(k) = p > 0$: rigid cohomology (Berthelot)
 - given an embedding $\sigma : k \subset \mathbb{C}$, the rational singular cohomology of the σ -complex points of a smooth k -scheme X ; this is called simply the rational Betti cohomology.
 - give a prime ℓ invertible in k , the \mathbb{Q}_ℓ -adic étale cohomology of $X \otimes_k \bar{k}$; this is called the geometric \mathbb{Q}_ℓ -adic cohomology.

We will denote by \mathbf{H}_ϵ the spectrum representing one of these Weil cohomologies: $\epsilon = \text{dR}$, rig , B , ℓ respectively. In all these cases, twists does not change the cohomology up to an isomorphism (see *loc. cit.* Introduction before theorem 1).

- (2) Note that Betti cohomology can be taken with integral coefficients. It is still representable in $\text{SH}(k)$, and twists do not change the cohomology (up to an isomorphism as above). We will denote by $\mathbf{H}_\sigma R$ the corresponding spectrum over k with coefficients in a ring R .

¹⁰This has been axiomatized in the notion of *mixed Weil cohomology theory* in [CD12].

- (3) Given now a real embedding $\sigma : k \subset \mathbb{R}$. One can consider the integral singular cohomology of the real points:

$$H^n(X^\sigma(\mathbb{R}), \mathbb{Z}).$$

This is representable by a ring spectrum that will be denoted by $\mathbf{H}_\sigma \mathbb{Z}$. In that case, twists just shift cohomology degrees, again up to isomorphisms:

$$(\mathbf{H}_\sigma \mathbb{Z})^{n,i}(X) = H^{n-i}(X^\sigma(\mathbb{R}), \mathbb{Z}).$$

Example 1.1.7. The following examples are absolute cohomology in the sense of Beilinson. In the motivic homotopy categorical sense, it means that the base scheme does not really matter when one computes the associated representable cohomology.¹¹

- (1) Motivic cohomology with coefficients in a ring R , $\mathbf{H}_{\mathbf{M},S}R$ can be defined for any scheme S .¹² The distinctive feature of motivic cohomology is:

$$\begin{aligned} \mathbf{H}_{\mathbf{M}}^{2n,n}(X, \mathbb{Z}) &= \mathrm{CH}^n(X) && X/S \text{ smooth, } S = \text{field, Dedekind ring} \\ \mathbf{H}_{\mathbf{M}}^{n,n}(k, \mathbb{Z}) &= \mathrm{K}_n^{\mathbf{M}}(k) && k \text{ any field.} \end{aligned}$$

where $\mathrm{CH}^n(X)$ denotes the Chow group of X : classes of codimension n algebraic cycles up for the rational equivalence, and $\mathrm{K}_n^{\mathbf{M}}(X)$ is the n -th Milnor K-group: the tensor algebra over the abelian group k^\times modulo the Steinberg relation.

- (2) Quillen algebraic K-theory over a regular scheme S . This is represented by a spectrum that we will denote by \mathbf{KGL}_S . This spectrum is periodic, in the sense that there exists a canonical isomorphism, the ‘‘Bott isomorphism’’:

$$\beta : \mathbf{KGL}_S(1)[2] \rightarrow \mathbf{KGL}_S.$$

Taken into account this isomorphism one gets the following distinctive property for any smooth S -scheme X :

$$\mathbf{KGL}^{n,i}(X) = K_{2i-n}(X)$$

where the right hand-side is Quillen algebraic K-theory: the $(2i - n)$ -th homotopy group of the nerve of the Q -category

¹¹Concretely, this means that these cohomologies are representable by an *absolute motivic spectrum*: a collection of motivic spectra \mathbb{E}_S over any scheme S , equipped for any morphism $f : T \rightarrow S$, with an isomorphism $f^*(\mathbb{E}_S) \simeq \mathbb{E}_T$ satisfying the usual cocycle condition. This is also a cartesian section of the fibred category SH over the category of schemes.

¹²The first definition of such a spectrum is due to Voevodsky. At the time being, one uses a definition based on higher Chow groups and due to Spitzweck. Both definitions coincide if S is smooth over a field.

associated with the exact category of vector bundles over X (pointed by the 0-object).

If one wants a true absolute spectrum, one will replace Quillen K-theory by Weibel homotopy invariant K-theory. We will still denote by \mathbf{KGL}_S the resulting spectrum, for an arbitrary base scheme S .

- (3) Algebraic cobordism over any base scheme S , denoted by \mathbf{MGL}_S . We will recall later the definition of this spectrum, and state a universal property.

1.1.8. Extended cohomology. Representable cohomology theory automatically acquire more structures. For once, a motivic spectrum \mathbb{E} defines a family of contravariant functors:

$$\tilde{\mathbb{E}}^{n,i} : (\mathbf{H}_{\bullet}^{\mathbb{A}^1}(S))^{op} \rightarrow \mathcal{A}b, \mathcal{X} \mapsto [\Sigma^{\infty} \mathcal{X}, \mathbb{E}(i)[n]].$$

Note an important property of this functor: it turns cofiber sequences in $\mathbf{H}_{\bullet}^{\mathbb{A}^1}(S)$ into long exact sequences of abelian groups.¹³

An interesting remark is that the \mathbb{E} -cohomology is therefore invariant under weak motivic equivalences.¹⁴

Secondly, one immediately gets a definition of cohomology with support. A closed S -pair (X, Z) will be a pair of schemes such that X is a smooth S -scheme, and $Z \subset X$ a closed subscheme. By taking homotopy cofibers, in the pointed motivic homotopy category, one can define the object $X/X - Z$ which fits into a cofiber sequence:

$$(X - Z)_+ \xrightarrow{j_*} X_+ \rightarrow X/X - Z$$

One defines the \mathbb{E} -cohomology of X with support in Z in degree n and twist i as:

$$\mathbb{E}_Z^{n,i}(X) := \tilde{\mathbb{E}}^{n,i}(X/X - Z).$$

Therefore, it fits into a long exact sequence:

$$\dots \mathbb{E}_Z^{n,i}(X) \rightarrow \mathbb{E}^{n,i}(X) \xrightarrow{j^*} \mathbb{E}^{n,i}(X - Z) \xrightarrow{\partial_{X,Z}} \mathbb{E}_Z^{n+1,i}(X) \dots$$

This cohomology with support enjoys good properties:

- (1) Contravariance: for any morphism $f : Y \rightarrow X$ of smooth S -schemes, there exists a pullback functor:

$$f^* : \mathbb{E}_Z^{n,i}(X) \rightarrow \mathbb{E}_{f^{-1}(Z)}^{n,i}(Y).$$

¹³This comes from the fact that Σ^{∞} , as a left adjoint, preserves cofiber sequences. As its target is a stable ∞ -category, it even sends cofiber sequences to exact sequences.

¹⁴To get a nice picture on weak motivic equivalences, we refer the survey paper [AOsr21].

- (2) Covariance: for any closed immersion $i : T \rightarrow Z$ of closed subschemes of X , one gets:

$$i_* : \mathbb{E}_T^{n,i}(X) \rightarrow \mathbb{E}_Z^{n,i}(X).$$

Remark 1.1.9. (1) All the previous examples admit a natural notion of cohomology with support, which agree with the above definition.

- (2) Morel-Voevodsky purity theorem implies the following property of cohomology with support, in the case where $Z \subset X$ is a smooth subscheme of codimension c :

$$\mathbb{E}_Z^{n,i}(X) \simeq \mathbb{E}^{n-2c,i-c}(Z).$$

1.1.c. **Ring spectra and cup-products.** The next definition is the last piece of structure one needs on cohomology to get characteristic classes.

Definition 1.1.10. A (*commutative*) *ring spectrum* \mathbb{E} over the base scheme S is a (commutative) monoid object in the homotopy category associated with $\mathrm{SH}(S)$.

In particular, the structure of a ring spectrum on \mathbb{E} is given by a unit $1_{\mathbb{E}} : \mathbb{1}_S \rightarrow \mathbb{E}$ and a product $\mu : \mathbb{E} \otimes_S \mathbb{E} \rightarrow \mathbb{E}$, which satisfies the usual axioms. If one wants to be precise, we will say that $(\mathbb{E}, \mu, 1_{\mathbb{E}})$ is a motivic ring spectrum.

One deduces a product on \mathbb{E} -cohomology, which is often called the cup-product¹⁵: given cohomology classes:

$$a : \Sigma^\infty X_+ \rightarrow \mathbb{E}(i)[n], b : \Sigma^\infty X_+ \rightarrow \mathbb{E}(j)[m]$$

one defines $a \cup_\mu b$ as the composite map:

$$\Sigma^\infty X_+ \xrightarrow{\delta_*} \Sigma^\infty (X \times_S X)_+ = \Sigma^\infty X_+ \otimes_S \Sigma^\infty X_+ \xrightarrow{a \otimes b} \mathbb{E} \otimes \mathbb{E}(i+j)[n+m] \xrightarrow{\mu} \mathbb{E}(i+j)[n+m].$$

We will usually denote this product simply as ab .

It follows that for any smooth S -scheme X , $\mathbb{E}^{**}(X)$ is a bi-graded algebra over the bigraded ring $\mathbb{E}^{**}(S)$, usually called the *coefficient ring* of \mathbb{E} and simply denoted by \mathbb{E}^{**} .

Remark 1.1.11. One should be careful that the above bigraded algebra is not simply graded commutative with respect to the first index. To state the required formula one needs the special element $\epsilon \in [\mathbb{1}_S, \mathbb{1}_S]$, which acts as a scalar on any representable cohomology theory, defined by the switch map inverse map $x \mapsto x^{-1}$ on \mathbb{G}_m , and using that $\Sigma^\infty(\mathbb{G}_m, 1) = \mathbb{1}_S(1)[1]$.

¹⁵This terminology, due to Whitney for the product on singular cohomology, has firmly remained in algebraic topology, due to the tremendous importance of its introduction in the thirties.

Then the ϵ -graded commutativity formula, for a, b as above, reads as follows:

$$(1.2) \quad ab = (-1)^{n+m-i-j} \cdot \epsilon^{i+j} \cdot ba$$

The proof is formal once one notices that ϵ can also be defined, up to \mathbb{A}^1 -homotopy, by the map switching the factors on $\mathbb{G}_m \times \mathbb{G}_m$ (see [Mor04, Lemma 6.1.1]).

Example 1.1.12. All the examples of cohomology theories of Example 1.1.6 and Example 1.1.7 are in fact representable by motivic ring spectra, and the associated cup-product corresponds to their usual product.

Remark 1.1.13. The theory developed below only requires the above definition. However, all the examples considered admits a *highly structured product*, *i.e.* it is the object in the homotopy category associated with a (commutative) algebra object of the monoidal ∞ -category $\mathrm{SH}(S)$. Beware that in general, it is fundamental in classical (and motivic) stable homotopy theory to give a clear distinction between those two kinds of structure.

1.1.d. Representability of the Picard group.

1.1.14. In algebraic topology, given an abstract group G , one can define its classifying space BG as an explicit simplicial set: the nerve of the groupoid associated with G , made of a single object $*$, a morphism for any element of g , the composition being given by the group law.

It is more common to consider topological groups (eg. Lie groups) G , and then one can still define a classifying space BG (as an explicit CW-complex) with the distinctive feature that for any topological space X , the homotopy classes of (unpointed) maps $[X, BG]$ are in bijection with the principal homogeneous G -spaces.

In motivic homotopy theory, Morel and Voevodsky have provided an analog of the second construction but using the first one and the framework of simplicial sheaves. For an algebraic group G over a scheme S , and a smooth S -scheme X , we denote by $H_{\mathrm{Nis}}^1(X, G)$ the set of G -torsors on X for the Nisnevich topology. Let us state a particular case of Morel-Voevodsky's construction relevant in our case.

Proposition 1.1.15. *Let S be a scheme, and G be an algebraic group over S .*

Then there exists an object BG in $\mathbf{H}_{\bullet}^{\mathbb{A}^1}(S)$ and for any smooth S -scheme X , a canonical functorial application of (pointed) sets:

$$(1.3) \quad H_{\mathrm{Nis}}^1(X, G) \rightarrow [X, BG]_S^{un}.$$

Moreover, if the left hand-side is \mathbb{A}^1 -invariant over all smooth S -schemes, this map is an isomorphism.

Note that it is very rare that the second condition holds. The only example we have in mind is that of $G = \mathbb{G}_m$ when S is regular. To sum-up in the case of \mathbb{G}_m , we get a canonical map:

$$\mathrm{Pic}(X) \rightarrow [X, B\mathbb{G}_{m,S}]_S^{un}$$

which is bijective whenever S is regular. Note also that the theory of Morel and Voevodsky shows that $B\mathbb{G}_{m,S}$ admits the geometric model that one expect: it is the infinite projective space \mathbb{P}_S^∞ , that is the infinite Grassmanian of lines in an affine space:

$$B\mathbb{G}_{m,S} = \varinjlim_{n \geq 0} \mathbb{P}_S^n$$

where the colimit can be taken in the category of simplicial sheaves on $\mathcal{S}m_S$ (to get an explicit model). Note that \mathbb{P}_S^∞ will be seen as a pointed sheaves via the point at ∞ of all the \mathbb{P}_S^n . We will recall from this discussion the canonical map:

$$(1.4) \quad \mathrm{Pic}(X) \rightarrow [X, \mathbb{P}_S^\infty]_S^{un}.$$

- Remark 1.1.16.* (1) Assume $S = \mathrm{Spec}(k)$ is the spectrum of a field. If one restricts our attention to smooth affine k -schemes X , then the map (1.3) is an isomorphism for an isotropic reductive k -group schemes: e.g. $G = \mathrm{GL}_n, \mathrm{SL}_n, \mathrm{Sp}_{2n}$. This is a theorem which was first obtained by Morel in certain cases, and in general by Asok, Hoyois and Wendt (see [AHW20] for the extra condition needed for G).
- (2) Morel and Voevodsky also give geometric models for over classifying spaces: as an example, for any $n \geq 0$, BGL_n is equivalent to the infinite grassmanian of sub- n -vector bundles:

$$\mathrm{BGL}_n = \varinjlim_{r \geq 0} \mathrm{Gr}(n, n+r).$$

1.2. ORIENTED RING SPECTRA

1.2.a. Definition and examples. Given the notation of the previous section, we have all the ingredients to formulate the notion of orientation, which has been introduced in motivic homotopy theory, by analogy with topology, at the time of the first proof of the Milnor conjecture by Voevodsky.

Definition 1.2.1. Let $(\mathbb{E}, \mu, 1_{\mathbb{E}})$ be a ring spectrum over S . Let $i : \mathbb{P}_S^1 \rightarrow \mathbb{P}_S^\infty$ be the canonical inclusion of pointed Nisnevich sheaves, both being pointed by the point at ∞ .

An *orientation* of \mathbb{E} is the data of a class $c \in \tilde{\mathbb{E}}^{2,1}(\mathbb{P}_S^\infty)$ such that $i^*(c) = 1_{\mathbb{E}}$ via the identification: $\tilde{\mathbb{E}}^{2,1}(\mathbb{P}_S^1) = \mathbb{E}^{0,0}(S)$.

We will say that the pair (\mathbb{E}, c) is an oriented (ring) spectrum.

Note that an orientation can be seen as a map:

$$c : \Sigma^\infty \mathbb{P}_S^\infty \rightarrow \mathbb{E}(1)[2].$$

Example 1.2.2. (1) Let X be a smooth \mathbb{Z} -scheme. We have seen that there is an isomorphism: $\mathbf{H}_M^{2,1}(X) \simeq \mathrm{CH}^1(X) \simeq \mathrm{Pic}(X)$. This extends to ind-smooth \mathbb{Z} -schemes. But $\mathrm{Pic}(\mathbb{P}_S^\infty) \simeq \mathbb{Z}\langle c \rangle$, the free abelian group generated by c , the class of the tautological invertible bundle $\lambda = \mathcal{O}_{\mathbb{P}_S^\infty}(-1)$.

Moreover, the restriction of c to \mathbb{P}_S^1 is the cycle class of the point at ∞ . It is the unit of the ring structure on $\mathrm{CH}^*(\mathbb{P}_S^1) \simeq \mathbf{H}_M^{2*,*}(\mathbb{P}_S^1) \simeq \mathbb{Z}\langle c \rangle$.¹⁶ Therefore, the class c corresponds to an orientation of $\mathbf{H}_M\mathbb{Z}$ over the base scheme \mathbb{Z} .

Given now any scheme X , we can look at the canonical map $f : S \rightarrow \mathbb{Z}$. Then f^* , being compatible with products on motivic cohomology, $f^*(c)$ is an orientation of $f^*\mathbf{H}_M\mathbb{Z} = \mathbf{H}_M\mathbb{Z}_S$.

- (2) Let \mathbf{H}_ϵ be the ring spectrum representing one of the mixed Weil cohomology theories, over smooth k -schemes with coefficient in the appropriate field K of characteristic 0 as in Example 1.1.6. The corresponding cohomology admits a cycle class map:

$$\mathrm{CH}^i(X) \simeq \mathbf{H}_M^{2i,i}(X) \rightarrow \mathbf{H}_\epsilon^{2i,i}(X) \xrightarrow{(*)} \mathbf{H}_\epsilon^{2i,0}(X) =: \mathbf{H}_\epsilon^{2i}(X)$$

which is compatible with products (mapping intersection products to “cup-products”).¹⁷ Therefore, the image of $c \in \mathrm{CH}^1(\mathbb{P}_k^\infty)$ in $\mathbf{H}_\epsilon^{2,1}(\mathbb{P}_S^\infty)$ induces a canonical orientation of the mixed Weil spectrum \mathbf{H}_ϵ .

- (3) The same strategy works for the singular cohomology of complex points of smooth algebraic k -schemes, over $k \subset \mathbb{C}$.
- (4) On the contrary, the ring spectrum representing singular cohomology of the real points of smooth algebraic k -scheme, $k \subset \mathbb{R}$, is not orientable. Indeed:

$$\mathbf{H}_\sigma^{2,1}(\mathbb{P}_k^1) = H^1(\mathbb{R}\mathbb{P}^1, \mathbb{Z}) = \mathbb{Z}$$

$$\mathbf{H}_\sigma^{2,1}(\mathbb{P}_k^\infty) = H^1(\mathbb{R}\mathbb{P}^\infty, \mathbb{Z}) = \mathbb{Z}/2.$$

¹⁶This follows for example from the definition via pullback along the diagonal.

¹⁷We put the last isomorphism to recall that mixed Weil cohomologies are $(0, 1)$ -periodic. This isomorphism is non-canonical and depends on the choice of a generator of the 1-dimensional K -vector space $\mathbf{H}_\epsilon^{1,1}(\mathbb{G}_{m,k})$. See [CD12] as indicated in Example 1.1.6.

- (5) The spectrum \mathbf{KGL} representing algebraic K -theory is oriented by the following class:

$$c^{\mathbf{KGL}}(L) = \beta^{-1}(1 - [\lambda^\vee]) \in \mathbf{KGL}^{2,1}(\mathbb{P}_S^\infty)$$

where we denote by $[L] \in \mathbf{KGL}^{0,0}(X) \simeq K_0(X)$ the class of a line bundle L/X in the Grothendieck group of vector bundles over X , and $\beta \in \mathbf{KGL}^{-2,-1}(S)$ is the Bott element (over S).

- (6) The algebraic cobordism spectrum \mathbf{MGL}_S admits, by construction, a canonical orientation. That will be cleared out in the next course.

1.2.b. Chern classes.

1.2.3. First Chern class. Let (\mathbb{E}, c) be an oriented ring spectrum over S . Taken into account the canonical map (1.4), we obtain for any smooth S -scheme X a canonical map:

$$\begin{aligned} \mathrm{Pic}(X) &\rightarrow [X_+, \mathbb{P}_S^\infty]_{\mathbb{H}\mathbb{A}^1(S)} \xrightarrow{\Sigma^\infty} [\Sigma^\infty X_+, \Sigma^\infty \mathbb{P}_S^\infty]_{\mathrm{SH}(S)} \\ &\xrightarrow{c^*} [\Sigma^\infty X_+, \mathbb{E}(1)[2]]_{\mathrm{SH}(S)} = \mathbb{E}^{2,1}(X). \end{aligned}$$

This is called the first Chern class associated with the orientation c , denoted simply by c_1 . It is clearly contravariantly functorial in the scheme X . However we must observe at this point that c_1 is simply an application, and not necessarily a morphism of groups. In fact, all the maps in the above compositum are morphisms of groups *except* the suspension map Σ^∞ . This fact is extremely meaningful in the theory of oriented ring spectra (see the next course).

The key fact of the theory is the following *projective bundle formula*:

Theorem 1.2.4. *Consider the above notation. Let $V \rightarrow X$ be a rank n vector bundle over a smooth S -scheme X , and let $P = \mathbb{P}(V)$ be the associated projective bundle. We let $p : P \rightarrow X$ be the canonical projection, and let λ_P be the canonical line bundle on P (coming from the fact $\mathbb{P}(V)$ classifies sub-line bundles of V).¹⁸ Then the following map:*

$$\begin{aligned} \bigoplus_{i=0}^{d-1} \mathbb{E}^{**}(X) &\rightarrow \mathbb{E}^{**}(P) \\ \lambda_i &\mapsto \sum_i p^*(\lambda_i) \cdot c_1(\lambda_P)^i \end{aligned}$$

*is an isomorphism of $\mathbb{E}^{**}(X)$ -modules.*

¹⁸This line bundle is often denoted by $\mathcal{O}_P(-1)$, for example by Fulton in [Ful98].

One can reformulate the above theorem by saying that $\mathbb{E}^{**}(P)$ is a bigraded $\mathbb{E}^{**}(X)$ -algebra (through the pullback map p^*) which is free of rank n , generated by $c_1(\lambda_P)^i$ for $0 \leq i \leq r - 1$.

Remark 1.2.5. Milnor sequence. In general, for a ring spectrum $(\mathbb{E}, \mu, 1_{\mathbb{E}})$ over S , one always has the so-called Milnor exact sequence:

$$0 \rightarrow \lim_{n \geq 0} {}^1\mathbb{E}^{2,1}(\mathbb{P}_S^n) \rightarrow \mathbb{E}^{2,1}(\mathbb{P}_S^\infty) \rightarrow \lim_{n \geq 0} \mathbb{E}^{2,1}(\mathbb{P}_S^n) \rightarrow 0.$$

It follows from the above that, whenever \mathbb{E} is oriented, the left hand side vanishes as the involved inductive system satisfies the Mittag-Leffler condition. In particular, to give an orientation on \mathbb{E} , it is sufficient to give classes $c_n \in \mathbb{E}^{2,1}(\mathbb{P}_S^n)$ for all $n > 0$ such that $c_1 = 1_{\mathbb{E}}$ and $\iota_n^*(c_{n+1}) = c_n$.

As a corollary, one gets our first family of characteristic classes, the Chern classes of algebraic vector bundles, following a method of Grothendieck.

Definition 1.2.6. Let (\mathbb{E}, c) be an oriented (motivic) ring spectrum over S . Let X be a smooth S -scheme and V/X be a vector bundle or rank n . Then there exists a unique family $(c_i(V))_{0 \leq i \leq n}$ such that the following relation holds in $\mathbb{E}^{2,1}(\mathbb{P}(V))$:

$$\sum_{i=0}^n p^*(c_i(V)) \cdot (-c_1(\lambda_P))^{n-i}$$

Note in particular that $c_i(V) \in \mathbb{E}^{2i,i}(X)$. If $i > n$, we put $c_i(V) = 0$.

1.2.7. According to the above definition, we get the following properties of Chern classes:

- (1) *Invariance under isomorphism.* For any isomorphism $V \simeq V'$ of vector bundles over X , $c_i(V) = c_i(V')$.
- (2) *Compatibility with pullbacks.* For any vector bundle V/X , and any morphism $f : Y \rightarrow X$ of smooth S -schemes, $f^*c_i(V) = c_i(f^{-1}V)$.
- (3) *Triviality.* For a trivializable vector bundle V , $c_i(V) = 0$ if $i > 0$.
- (4) *Nilpotence.* Here it is important that S is noetherian. For any vector bundle V/X , and any $i \geq 0$, the Chern class $c_i(V)$ is nilpotent.

The third relation follows from the fact $c_1(\mathcal{O}_{\mathbb{P}^n}(-1))^{n+1} = 0$ (see the proof of the projective bundle theorem). The last relation is left as an exercise to the reader.

To go further, one needs the so-called *splitting principle*. It is based on the following “splitting construction”.

Proposition 1.2.8. *Let X be a smooth S -scheme, and V a vector bundle over X . Then there exists a smooth projective map $p : X' \rightarrow X$ such that $p^{-1}(E)$ splits as a direct sum of line bundles and such that for any oriented ring spectrum \mathbb{E} over S , the pullback map $p^* : E^{**}(X) \rightarrow E^{**}(X')$ is injective.*

Remark 1.2.9. A canonical construction for X' is to take the flag bundle associated with V , which is the moduli space which parametrize the complete flag of sub-vector bundles of V . The fact the projection map induces an injective pullback on an oriented cohomology theory can be seen as a *motivic Leray-Hirsch theorem*. The latter can be obtained directly from the homotopy Leray spectral sequence of [ADN20] associated with p and with coefficients in \mathbb{E} .

1.2.10. Splitting principle. As a corollary of the previous proposition, one obtains the so-called splitting principle for Chern classes associated with any oriented ring spectrum (\mathbb{E}, c) as above. Let V/X be a rank n vector bundle over a smooth S -scheme X .

First, we define the total Chern class as the polynomial in t , with coefficients in the (bigraded) ring $\mathbb{E}^{**}(X)$:¹⁹

$$c_t(V) = \sum_{i \geq 0} c_i(V) \cdot t^i.$$

Then the splitting principle tells us that, to compute with the Chern classes of V , one can assume that V is split using the preceding splitting construction. this amounts to say that the total Chern class splits: it admits Chern roots α_i such that:

$$c_t(V) = \prod_{i=1}^n (1 + \alpha_i \cdot t)$$

Then any symmetric polynomial in the Chern roots α_i admits an expression in terms of the Chern classes of V .

As an example, one can get the formula:

Proposition 1.2.11 (Whitney sum formula). *For any exact sequence of vector bundles over a smooth S -scheme X :*

$$0 \rightarrow V' \rightarrow V \rightarrow V'' \rightarrow 0$$

¹⁹This convention for total Chern class follows Fulton [Ful98]. Other conventions, such as for example in [MS74] simply considers the sum $c(V) = \sum_i c_i(V)$ in the “total” cohomology $\bigoplus_i \mathbb{E}^{2i,i}(X)$.

one has: $c_t(V) = c_t(V') \cdot c_t(V'')$.

Example 1.2.12. Consider the above notation. Given V/X a vector bundle of rank n , one usually defines the *Euler class* of V as:

$$e(V) = c_n(V).$$

Assume X is smooth affine of dimension n over $S = \text{Spec}(k)$, the spectrum of an algebraically closed field (or a field in which (-1) is a sum of squares). Then we have seen in the talk of Aravind Asok that the vanishing of the Euler class in motivic cohomology, is equivalent to the fact V splits-off a trivial summand. (One direction obviously follows from the above Whitney sum formula!)

However, to remove the assumption on k , one needs a finer version of the Euler class, with values in the Chow-Witt group.

1.2.c. The algebraic Hopf map.

1.2.13. The endomorphism ring of the sphere spectrum, $\text{End}(\mathbb{1}_S)$ acts on any motivic spectrum \mathbb{E} . Similarly, any map $\varphi : \mathbb{1}_S \rightarrow \mathbb{1}_S(i)[n]$ induces a morphism $\varphi \otimes \mathbb{E} : \mathbb{E} \rightarrow \mathbb{E}(i)[n]$. This can be seen as an action of the graded ring $\Pi^{n,i}(S)$ — the stable motivic cohomotopy of S — on \mathbb{E} .

According to the fundamental theorem of Morel, when S is the spectrum of a field k , one gets $\Pi^{n,n}(k) \simeq K_n^{\text{MW}}(k)$, the Milnor-Witt ring of k . Other any base S , one still gets important endomorphisms:

- (1) *Algebraic Hopf map.* $\eta : \mathbb{1}_S(1)[1] \rightarrow \mathbb{1}_S$, which is induced by the canonical map $\mathbb{A}_S^2 - \{0\} \rightarrow \mathbb{P}_S^1$, $(x, y) \mapsto [x : y]$ (in coordinates).
- (2) *Classes of units.* for any $u \in \mathcal{O}(S)^\times$, one deduces $[u] : \mathbb{1}_S \rightarrow \mathbb{1}_S(1)[1]$ from the map $u : S \rightarrow \mathbb{G}_{m,S}$ corresponding to u . One then puts:

$$\langle u \rangle = 1 + \eta \cdot [u],$$

which is an element in degree $(0, 0)$ of the bigraded ring $\Pi^{**}(S)$.

Note that one can check that $\epsilon = - \langle -1 \rangle \in \Pi^{0,0}(S)$, where ϵ was defined in Remark 1.1.11.²⁰

As a consequence of the projective bundle theorem and using the above mentioned remark, one deduces:

Proposition 1.2.14. *Let \mathbb{E} be an orientable ring spectrum. Then the algebraic Hopf map η acts trivially on \mathbb{E} : $\eta \otimes \mathbb{E} = 0$.*

²⁰In fact, η and ϵ are defined by pullbacks from elements of $\Pi^{**}(\mathbb{Z})$. It is likely that $\Pi^{0,0}(\mathbb{Z}) = \mathbb{Z}[\epsilon]/(\epsilon^2 = 1)$. This would be a direct consequence of the absolute purity property for (reduced) closed subschemes of $\text{Spec}(\mathbb{Z})$.

As a consequence, for every units $u \in \mathcal{O}(S)^\times$, $\langle u \rangle$ acts by the identity. In particular, ϵ acts by (-1) : $\epsilon \otimes \mathbb{E} = -\text{Id}_{\mathbb{E}}$. As a consequence, relation (1.2) becomes

$$ab = (-1)^{nm}.ba.$$

Proof. The first assertion follows from the cofiber sequence in the pointed motivic homotopy category:

$$\mathbb{A}^2 - \{0\} \xrightarrow{\eta} \mathbb{P}_S^1 \xrightarrow{\iota_1} \mathbb{P}_S^2$$

for which we refer to [Mor04]. Indeed, if \mathbb{E} is oriented, then $\mathbb{E} \otimes \iota_1$ is a split monomorphism. The rest of the assertions follow easily. \square

In general, the action of the Hopf map is not sufficient to detect orientability of a ring spectrum. However, we have the notable theorems.

Theorem 1.2.15. *Let k be a perfect field, and $\mathbb{E} \in \text{SH}(k)$ be a homotopy module with a ring structure. Then the following conditions are equivalent:*

- (i) \mathbb{E} is orientable.
- (ii) $\eta \otimes \mathbb{E} = 0$.
- (iii) \mathbb{E} admits transfers in the sense of Voevodsky (i.e. action of finite correspondences).

This theorem uses the equivalence between homotopy modules with transfers and Rost cycle modules: see [Dég13]. We can now obtain a more direct proof by using the equivalence of homotopy modules with Milnor-Witt cycle modules: see [Fel21].

Theorem 1.2.16 (Morel, Cisinski-D.). *Let \mathbb{E} be a rational motivic ring spectrum over a scheme S . Then the following conditions are equivalent:*

- (i) \mathbb{E} is orientable.
- (ii) $\eta \otimes \mathbb{E} = 0$.
- (iii) $\epsilon \otimes \mathbb{E} = -\text{Id}_{\mathbb{E}}$.

In fact in these case, \mathbb{E} is a rational motive !

Sketch of proof.²¹ The proof relies on Morel's decomposition of the rational stable homotopy category into:

$$\text{SH}(S)_{\mathbb{Q}} \simeq \text{SH}(S)_{\mathbb{Q}^+} \times \text{SH}(S)_{\mathbb{Q}^-}$$

characterized by the equivalent properties:

- (i) $\mathbb{E} \in \text{SH}(S)_{\mathbb{Q}^+}$ (resp. $\mathbb{E} \in \text{SH}(S)_{\mathbb{Q}^-}$).
- (ii) $\epsilon \otimes \mathbb{E}$ is equal to -1 (resp. $+1$).

²¹This proof is a simplification of the proof given in [CD19, Th. 16.2.13].

(iii) $\eta \otimes \mathbb{E}$ is null (resp. invertible).

Then the main point is to show that the canonical map:

$$\mathbb{1}_S \otimes \mathbb{Q}_+ \rightarrow \mathbf{H}_{\mathbb{E}, S}$$

is an isomorphism, where the right hand-side is Beilinson motivic cohomology ring spectrum (representing the 0-th graded piece of rational algebraic K -theory, over regular schemes). By a localization arguments and invariance under inseparable field extensions, one reduces to the case of a perfect field k . Then a devissage argument (we use rational coefficients at this point) reduces to the preceding theorem.

Remark 1.2.17. As a complement, let us say that one now knows how to compute both the plus and the minus part of rational motivic stable homotopy category (see [DFJK21]):

$$\mathrm{SH}(S)_{\mathbb{Q}_+} \simeq \mathrm{DM}(S, \mathbb{Q})$$

is the category of rational mixed motivic complexes. In particular, rationally, being orientable is the same as being a motivic complex.

For the minus part, one has:

$$\mathrm{SH}(S)_{\mathbb{Q}_-} \simeq \mathbf{H} \underline{\mathbf{W}}_{S \otimes_{\mathbb{Z}} \mathbb{Q}} - \mathrm{mod}$$

where the right hand-side is the category of modules over the unramified rational Witt sheaf, seen over the characteristic 0 part $S \otimes_{\mathbb{Z}} \mathbb{Q}$ of S (in particular, it is zero on a scheme of positive charadteristic).

Cours 2. Oriented spectra: Thom classes and formal group laws

2.1. THOM CLASSES

2.1.a. Construction.

2.1.1. Let V be a rank n vector bundle over a smooth S -scheme X , and (\mathbb{E}, c) be an oriented ring spectrum over S . We let $\nu : \mathbb{P}(V) \rightarrow \mathbb{P}(V \oplus \mathbb{A}^1)$ be the canonical closed immersion.²²

Recall from the talk of Kirsten Wickelgren that one defines the Thom space of V in the pointed motivic homotopy category $\mathbf{H}_{\bullet}^{\mathbb{A}^1}(S)$ over S as the following homotopy cofibers:

$$\mathrm{Th}_S(V) = V/V^\times = \mathbb{P}(V \oplus \mathbb{A}^1)/\mathbb{P}(V).$$

²²The target of ν is known as the projective completion of V : the open complement of ν is isomorphic to V .

In general, it is clear over which base one considers the Thom space $\mathrm{Th}_S(V)$ so that we will denote it simply by $\mathrm{Th}(V)$.²³

One deduces from the above formula for Thom spaces a long exact sequence

$$(2.5) \quad \dots \rightarrow \tilde{\mathbb{E}}^{**}(\mathrm{Th}(V)) \xrightarrow{\pi^*} \mathbb{E}^{**}(\mathbb{P}(V \oplus \mathbb{A}^1)) \xrightarrow{\nu^*} \mathbb{E}^{**}(\mathbb{P}(V)) \rightarrow \dots$$

where we have used the extension of \mathbb{E}^{**} described in 1.1.8.²⁴

It follows from Theorem 1.2.4 that the map ν^* is a split epimorphism of free $\mathbb{E}^{**}(X)$ -modules of respective ranks n and $n - 1$. Thus $\mathbb{E}^{**}(\mathrm{Th}(X))$ is a free $\mathbb{E}^{**}(X)$ -module of rank 1, isomorphic to $\ker(\nu^*)$. One deduces from this discussion the following definition.

Definition 2.1.2. Consider the above notation and assumptions.

We define the *Thom class* of V/X as the following element of $\mathbb{E}^{2n,n}(\mathbb{P}(V \oplus \mathbb{A}^1))$:

$$\mathrm{th}(V) = \sum_{i=0}^n p^*(c_i(V)) \cdot (-c_1(\lambda))^{n-i}$$

using the notation of Theorem 1.2.4. We define the *refined Thom class* $\overline{\mathrm{th}}(V)$ of V as the unique element of $\mathbb{E}^{2n,n}(\mathrm{Th}(V))$ such that

$$\pi^*(\overline{\mathrm{th}}(V)) = \mathrm{th}(V).$$

It follows from the split exact sequence (2.5) of $\mathbb{E}^{**}(X)$ -modules that the following map

$$(2.6) \quad \mathbb{E}^{**}(X) \rightarrow \mathbb{E}^{**}(\mathrm{Th}(V)), \lambda \mapsto \lambda \cdot \overline{\mathrm{th}}(V)$$

is an isomorphism of bidegree $(2n, n)$, called the *Thom isomorphism* associated with the vector bundle V/X and with coefficients in the oriented ring spectrum (\mathbb{E}, c) .

We deduce from the analogous properties of Chern classes that Thom classes are compatible with base change and invariant under isomorphisms of vector bundles.

Example 2.1.3. Recall the universal quotient bundle ξ on $\mathbb{P}(V \oplus \mathbb{A}^1)$ is defined by the exact sequence

$$0 \rightarrow \mathcal{O}(-1) \rightarrow p^{-1}(V \oplus 1) \rightarrow \xi \rightarrow 0.$$

²³Beware however it could also be considered as an object of $H_{\bullet}^{\mathbb{A}^1}(X)$, which is reflected in the more precise notation $\mathrm{Th}_X(V)$.

²⁴Recall that $\tilde{\mathbb{E}}^{**}(\mathrm{Th}(V))$ can also be described as the \mathbb{E} -cohomology of V with support in the 0-section of V/X .

Thus the Whitney sum formula Proposition 1.2.11 gives the following relation between Thom and Chern classes:

$$(2.7) \quad \text{th}(V) = c_n(\xi) = e(\xi).$$

2.1.4. It will be useful to work internally, and relatively. Consider the notation of the previous definition and let $p : X \rightarrow S$ be a smooth morphism.

Recall we have the base change functor $p^* : \text{SH}(S) \rightarrow \text{SH}(X)$ with left adjoint $p_{\#}$. We put $\mathbb{E}_X = p^*(\mathbb{E})$. Note the orientation of \mathbb{E} determines a canonical orientation of \mathbb{E}_X . The object \mathbb{E}_X is a commutative monoid of $\text{HoSH}(X)$ so we can consider the additive category of \mathbb{E}_X -modules. Then the Thom isomorphism associated with a vector bundle V of rank r over X actually corresponds to an isomorphism of \mathbb{E}_X -modules:

$$\text{th}(V) : \mathbb{E}_X \otimes \Sigma^\infty \text{Th}(V) \rightarrow \mathbb{E}_X(r)[2r].$$

2.1.b. Universal property of algebraic cobordism.

2.1.5. *Construction of algebraic cobordism.* Let us now recall the construction of the algebraic cobordism spectrum \mathbf{MGL}_S (which is modeled on the construction of the topological spectra MU , MO , ...).

Though all objects will be considered over S , we drop the index in the notation for simplicity. We consider the tautological rank n vector bundle γ_n on the classifying space BGL_n viewed as a smooth S -scheme via the model given by the infinite grassmannian:

$$\text{BGL}_n = \text{colim}_{m \geq n} \text{Gr}_n(\mathbb{A}^m).$$

As $\gamma_n \oplus \mathbb{A}^r$ as rank $n+r$, one deduces a (homotopy) cartesian square:

$$\begin{array}{ccc} \gamma_n \oplus \mathbb{A}^r & \xrightarrow{(*)} & \gamma_{n+r} \\ \downarrow & & \downarrow \\ \text{BGL}_n & \longrightarrow & \text{BGL}_{n+r}. \end{array}$$

One deduces from (*) a canonical map of Thom spaces in $\mathbf{H}_\bullet^{\mathbb{A}^1}(S)$:

$$\text{Th}(\gamma_n)(r)[2r] = \text{Th}(\gamma_n \oplus \mathbb{A}^r) \rightarrow \text{Th}(\gamma_{n+r}).$$

In particular, one deduces a tower in $\text{SH}(S)$:

$$\begin{aligned} \Sigma^\infty \text{Th}(\gamma_0) &\rightarrow \Sigma^\infty \text{Th}(\gamma_1)(-1)[-2] \rightarrow \\ &\dots \rightarrow \Sigma^\infty \text{Th}(\gamma_n)(-n)[-2n] \rightarrow \dots \end{aligned}$$

One defines the algebraic cobordism spectrum as the homotopy colimit:

$$(2.8) \quad \mathbf{MGL}_S = \text{hocolim}_{n \geq 0} \Sigma^\infty \text{Th}_S(\gamma_n)(-n)[-2n].$$

Further, using the canonical map $\gamma_n \times \gamma_m \rightarrow \gamma_{n+m}$ one can build ring structure on \mathbf{MGL}_S .²⁵

Note that the vector bundle γ_1 over $B\mathbb{G}_{m,S}$ can be identified with the canonical line bundle $\mathcal{O}_{\mathbb{P}^\infty}(-1)$ on \mathbb{P}^∞ via the weak \mathbb{A}^1 -equivalence $B\mathbb{G}_{m,S} \simeq \mathbb{P}_S^\infty$. We need the following lemma.

Lemma 2.1.6. *Consider the above notion. Then there exists a canonical weak \mathbb{A}^1 -homotopy equivalence of pointed motivic spaces over S :*

$$\mathbb{P}_S^\infty \simeq \mathrm{Th}(\gamma_1).$$

Proof. We consider the closed immersion:

$$\mathbb{P}_S^{n-1} \xrightarrow{\iota_n} \mathbb{P}_S^n.$$

The normal bundle of ι_n is the canonical line bundle $\mathcal{O}_{\mathbb{P}^{n-1}}(-1)$ on \mathbb{P}^{n-1} . Moreover, the open complementary $\mathbb{P}_S^n - \mathbb{P}_S^{n-1}$ is isomorphic to the affine line \mathbb{A}_S^n , so it is contractible. Using Morel-Voevodsky's purity theorem, one gets:

$$(\mathbb{P}_S^n, 1) \simeq \mathbb{P}_S^n / (\mathbb{P}_S^n - \mathbb{P}_S^{n-1}) \simeq \mathrm{Th}(\mathcal{O}_{\mathbb{P}^{n-1}}(-1)).$$

It is important to note that this isomorphism is functorial with respect to the inclusion ι_n . Therefore, one can take the homotopy limit over n , and this defines the required isomorphism. \square

2.1.7. Canonical orientation of \mathbf{MGL}_S . One deduces from the above lemma a canonical map:

$$c^{\mathbf{MGL}} : \Sigma^\infty \mathbb{P}_S^\infty \rightarrow \Sigma^\infty \mathrm{Th}(\gamma_1) \rightarrow \mathbf{MGL}_S(1)[2].$$

By construction (see the above proof for $n = 1$), the restriction of this map to (\mathbb{P}_S^1, ∞) corresponds up to \mathbb{P}^1 -desuspension to the unit:

$$\Sigma^\infty S_+ = \Sigma^\infty \mathrm{Th}(\gamma_0) \rightarrow \mathbf{MGL}_S$$

Therefore, $c^{\mathbf{MGL}}$ is an orientation of \mathbf{MGL}_S .

The main theorem of this section is the following universality theorem:

Theorem 2.1.8 (Vezzosi, Panin-Pimenov-Röndigs, Naumann-Østvær-Spitzweck). *Let \mathbb{E} be a ring spectrum over S . Then the following sets are in bijective correspondence:*

- (i) orientations c of \mathbb{E} ;
- (ii) morphisms of ring spectra $\varphi : \mathbf{MGL}_S \rightarrow \mathbb{E}$.

²⁵In fact, Tom Bachmann and Marc Hoyois have shown in [BH21, 16.2] how to give an E_∞ -ring structure on \mathbf{MGL}_S using the so-called motivic J-homomorphism, which is an ∞ -categorical enhancement of the stable Thom space functor.

by the map:

$$(2.9) \quad (ii) \rightarrow (i), \varphi \mapsto \varphi_*(c^{\mathbf{MGL}})$$

where $\varphi_* : \mathbf{MGL}^{**} \rightarrow \mathbb{E}^{**}$ is the induced map on cohomology.

In other words, \mathbf{MGL}_S is the universal oriented ring spectrum over S .

Idea of Proof. There are several steps for this proof, which works as in topology.

The first step is to determine the \mathbb{E} -cohomology of $\mathrm{BGL}_\infty = \mathrm{colim}_n \mathrm{BGL}_n$, for (\mathbb{E}, c) oriented. As in topology, one gets:

$$\mathbb{E}^{**}(\mathrm{BGL}_\infty) \simeq \mathbb{E}^{**}(S)[[c_1, c_2, \dots]]$$

where c_n is the n -th Chern class of the tautological rang n bundle on BGL_n . Note in passing that the preceding computation uses the Milnor exact sequence:

$$0 \rightarrow \lim_{n \geq 0}^1 \mathbb{E}^{**}(\mathrm{BGL}_n) \rightarrow \mathbb{E}^{**}(\mathrm{BGL}_\infty) \rightarrow \lim_{n \geq 0} \mathbb{E}^{**}(\mathrm{BGL}_n) \rightarrow 0$$

and the vanishing of the first term (as in Remark 1.2.5).

One deduces, from a similar Milnor exact sequence and formula (2.8) that the following canonical maps are isomorphism:

$$\begin{aligned} \mathbb{E}^{**}(\mathbf{MGL}_S) &\rightarrow \lim_{n \geq 0} \mathbb{E}^{**+2n, *+n}(\mathrm{Th}(\gamma_n)) \\ &\xrightarrow{\mathrm{th}} \lim_{n \geq 0} \mathbb{E}^{**}(\mathrm{BGL}_n) \cdot \mathrm{th}(\gamma_n) \simeq \mathbb{E}^{**}(\mathrm{BGL}_\infty), \end{aligned}$$

where th is given by the Thom isomorphisms of γ_n constructed previously.

It follows that the sequence $(\mathrm{th}(\gamma_0), \mathrm{th}(\gamma_1), \dots)$ uniquely defines an element $\mathrm{th}^c \in \mathbb{E}^{00}(\mathbf{MGL}_S)$, canonically associated to the orientation c . It remains to prove that $\mathrm{th}_c : \mathbf{MGL}_S \rightarrow \mathbb{E}$ is in fact a morphism of ring spectra. And moreover, the application $c \mapsto \mathrm{th}^c$ is a left and right inverse to (2.9). \square

Remark 2.1.9. This result was first proved over a field by Vezzosi in [Vez01]. It was later revisited in [PPR08], still over a field. Both proofs were in fact valid over a general base. A definitive reference, valid over an arbitrary base was finally given in [NSØ09], partly based on the Landweber exactness theorem.

2.1.c. Virtual Thom classes.

2.1.10. The Thom classes associated with on oriented ring spectrum (\mathbb{E}, c) are multiplicative. Let $p : X \rightarrow S$ be a smooth morphism and consider an exact sequence of vector bundles over X :

$$(2.10) \quad 0 \rightarrow V' \rightarrow V \rightarrow V'' \rightarrow 0.$$

One deduces from this exact sequence a canonical isomorphism in $\mathrm{SH}(X)$:

$$(2.11) \quad \Sigma^\infty \mathrm{Th}(V) \simeq \Sigma^\infty \mathrm{Th}(V') \otimes_X \Sigma^\infty \mathrm{Th}(V'').$$

Further:

Lemma 2.1.11. *Consider the above assumptions. Then, the isomorphism (2.11) induces an identification:*

$$\mathbb{E}^{**}(\mathrm{Th}(V)) \simeq \mathbb{E}^{**}(\mathrm{Th}(V')) \otimes_{\mathbb{E}^{**}(X)} \mathbb{E}^{**}(\mathrm{Th}(V'')),$$

and through this identification, one has: $\mathrm{th}(V) = \mathrm{th}(V') \otimes \mathrm{th}(V'')$.

Proof. One reduces to the case where the sequence is split. Then we have an isomorphism of vector bundles over $\mathbb{P}(V)$:

$$\xi = p^{-1}(V)/\mathcal{O}(-1) \simeq p^{-1}(V')/\mathcal{O}(-1) \oplus p^{-1}(V'')/\mathcal{O}(-1) = \xi' \oplus \xi''.$$

According to the Whitney sum formula, one deduces $e(\xi) = e(\xi').e(\xi'')$. One concludes using the fact $\mathbb{P}(V) - \mathbb{P}(V')$ is a vector bundle over $\mathbb{P}(V'')$. \square

2.1.12. We can elaborate on the previous result as follows. Consider the previous notation.

We let $\underline{\mathbf{K}}(X)$ be the Picard 1-category of virtual vector bundle.²⁶ One deduces from the isomorphisms of the form (2.11) that the Thom spectrum functors extends to a *virtual Thom spectrum functor*²⁷:

$$\underline{\mathbf{K}}(X) \rightarrow \mathrm{Ho} \mathrm{SH}(X), v \mapsto \mathrm{Th}(v).$$

The cohomology with coefficients in \mathbb{E} can be defined on these virtual Thom spectra:

$$\mathbb{E}^{n,i}(\mathrm{Th}(v)) = [\mathrm{Th}(v), \mathbb{E}_X(i)[n]]_X.$$

There is an obvious $\mathbb{E}^{**}(X)$ -module structure on these groups. Moreover, the preceding lemma shows that these $\mathbb{E}^{**}(X)$ -modules are free of

²⁶This is the groupoid associated with Quillen K -theory space $\mathbf{K}(X)$. A direct construction is given in [Del87].

²⁷One can deduce it by elementary means, but we now know that this is actually the truncation of the motivic J -homomorphism defined by Tom Bachmann and Marc Hoyois: $J : \mathbf{K}(X) \rightarrow \mathrm{Pic}(\mathrm{SH}(X))$. See [BH21, §16.2].

rank 1, and moreover, admits a canonical $\mathbb{E}^{**}(X)$ -basis $\text{th}(v)$ of bidegree $(2r, r)$, where $r \in \mathbb{Z}$ is the rank of v .²⁸

Definition 2.1.13. Given a virtual vector bundle v over a smooth S -scheme X , of virtual rank $r \in \mathbb{Z}$, we let $\text{th}(v) \in \mathbb{E}^{2r,r}(\text{Th}(v))$ be the class defined above.

By definition, the following morphism:

$$\mathbb{E}^{**}(X) \rightarrow \mathbb{E}^{**}(\text{Th}(v)), \alpha \mapsto \alpha \cdot \text{th}(v).$$

is an isomorphism of $\mathbb{E}^{**}(X)$ -modules, called the Thom isomorphism of v associated with the orientation c on \mathbb{E} .

Note finally that, as in 2.1.4, one deduces a isomorphism of \mathbb{E}_X -modules:

$$(2.12) \quad \text{th}(v) : \mathbb{E}_X \otimes \text{Th}(v) \xrightarrow{\sim} \mathbb{E}_X(r)[2r].$$

2.2. FORMAL GROUP LAWS AND ORIENTATIONS

2.2.a. Recall on formal group laws.

2.2.1. From the point of view of algebraic geometry, a *commutative formal group law of dimension 1* over a ring R is an *abelian group object structure* on the formal scheme $\text{Spf}(R[[x]])$, in the category of formal schemes.²⁹ We will say FGL (over R) for abelian formal group law of dimension 1.

This is equivalent to the (usual) concrete definition: such an FGL is given by a power series $F(x, y) \in R[[x, y]]$ satisfying the properties:

- (1) *Neutral element.* $F(x, 0) = x$.
- (2) *Commutativity.* $F(x, y) = F(y, x)$.
- (3) *Associativity.* $F(x, F(y, z)) = (F(x, F(y, z)))$.³⁰

(Recall that the existence of the formal inverse follows from these conditions; see e.g. [Str19, Lem. 2.7].) We will consider the following generic form for such an FGL:

$$F(x, y) = x + y + \sum_{i,j} a_{ij} \cdot x^i y^j.$$

²⁸More precisely, one uses the universal property of the Picard category $\underline{\mathbf{K}}(X)$ and the preceding lemma to get both results.

²⁹We see $R[[x]]$ as an admissible ring via the ideal of definition (x) . Therefore $\text{Spf}(R[[x]])$ is nothing else than the topological space $\text{Spec}(R)$ seen as a ring space via the sheaf associated to the pro-ring $R[[x]]$.

³⁰Note that the substitutions are licite because property (1) implies that $F(x, y)$ has no constant term.

Example 2.2.2. Recall that the only examples of formal group laws such that $F(x, y)$ has only finitely many coefficients are:

- (1) *Additive FGL*: $F_{add}(x, y) = x + y$.
- (2) *u -Multiplicative FGL*: $F_{mul,u}(x, y) = x + y + u.xy$, for $u \in R$.

Remark 2.2.3. FGL can be based changed: given a morphism of rings $\varphi : R \rightarrow R'$, and an FGL $F(x, y)$ over R , we obtain an FGL $F_{R'}(x, y)$ over R' by applying φ coefficient-wise.

2.2.4. Let R be a ring, and $F(x, y), G(x, y)$ two FGL with coefficients in R . An isomorphism θ from $F(x, y)$ to $G(x, y)$ will be a power series $\theta(t) = a_1.t + a_2.t^2 + \dots$ such that $a_1 \in R^\times$, and the following relation holds in $R[[x, y]]$:

$$\theta(F(x, y)) = G(\theta(x), \theta(y)).$$

When one considers $(R, F(x, y))$ and $(R', G(x, y))$ two FGL with different ring of coefficients, one defines a morphism from the first to the second as a pair (φ, θ) such that $\varphi : R \rightarrow R'$ is a morphism of rings, and $\theta : F_{R'}(x, y) \rightarrow G(x, y)$ is an isomorphism of FGL over R' . One deduces a fibered category \mathcal{FGL} over rings whose objects are pairs $(R, F(x, y))$ and morphisms are described as above.

Let us recall that the theorem of Lazard asserts that \mathcal{FGL} admits an initial object $(\mathbb{L}, F_{\mathbb{L}})$ such that $\mathbb{L} = \mathbb{Z}[a_1, a_2, \dots]$ is a polynomial ring, now called the *Lazard ring*.

2.2.b. Orientations and FGL.

2.2.5. We fix an oriented motivic ring spectrum (\mathbb{E}, c) over S (Definition 1.2.1), with ring of coefficients $\mathbb{E}^{**} = \mathbb{E}^{**}(S)$.

As the group scheme \mathbb{G}_m is abelian, one immediately deduces that $B\mathbb{G}_m$ is an h-group *i.e.* a group object in the homotopy category of $\mathrm{Sh}^\infty(\mathcal{S}m_S)$, and therefore also after applying the \mathbb{A}^1 -localization functor.

On the other hand \mathbb{P}_S^∞ , being the moduli space of line bundles (relative to S), it automatically acquires a structure of an abelian group object in the category of ind-smooth S -schemes, corresponding to the existence of tensor product of line bundles, as well as the inverse functor $\lambda \mapsto \lambda^\vee$. In particular, we get a canonical map:

$$\sigma : \mathbb{P}_S^\infty \times_S \mathbb{P}_S^\infty \rightarrow \mathbb{P}_S^\infty$$

The reader can check that it is actually given by the classical Segre embedding.

Consider an oriented ring spectrum (\mathbb{E}, c) . The projective bundle formula Theorem 1.2.4 implies that \mathbb{E}^{**} satisfies the Künneth formula on product of projective spaces. This, together with the preceding

observation, readily implies that $\mathbb{E}^{**}(\mathbb{P}_S^\infty)$ is an abelian co-group object. In other words, abelian group structure on \mathbb{P}_S^∞ determines an FGL with coefficients in $\mathbb{E}^{**}(S)$: one has a canonical map:

$$\mathbb{E}^{**}[[c]] \simeq \mathbb{E}^{**}(\mathbb{P}_S^\infty) \xrightarrow{\sigma^*} \mathbb{E}^{**}(\mathbb{P}_S^\infty \times_S \mathbb{P}_S^\infty) \simeq \mathbb{E}^{**}[[x, y]]$$

and one defines $F_c(x, y)$ as the image of c .

Definition 2.2.6. Given the above notation, we will say that $F_c(x, y)$ is the *formal group law* associated with the oriented ring spectrum (\mathbb{E}, c) .

We will say that (\mathbb{E}, c) (or just c) is respectively *additive* or *multiplicative* (with parameter u) if $F_c(x, y)$ is the additive or (u -)multiplicative) FGL.

There is a constraint on the coefficients say a_{ij}^S of $F_c(x, y)$ coming from the fact \mathbb{E}^{**} is bigraded: in fact a_{ij}^S has bidegree $(2-2i-2j, 1-i-j)$ in \mathbb{E}^{**} .

Note that by base change along $p^* : \mathbb{E}^{**}(X) \rightarrow \mathbb{E}^{**}(S) = \mathbb{E}^{**}$, for any smooth map $p : X \rightarrow S$, we deduce from the FGL $F_c(x, y)$ over \mathbb{E}^{**} and FGL over $\mathbb{E}^{**}(X)$ that we will denote by $F_c^X(x, y)$ (or simply $F_c(x, y)$ when X is clear).

Before giving examples, let us explain what is the concrete signification of the above FGL in term of characteristic classes.

Proposition 2.2.7. *Consider the notations of the previous definition.*

Then for any line bundles L_1, L_2 over X , one has the following relation in $\mathbb{E}^{2,1}(X)$

$$c_1(L_1 \otimes L_2) = F_c^X(c_1(L_1), c_1(L_2)).$$

Note in particular that this relation makes sense because the classes $c_1(L)$ and $c_1(L')$ are nilpotent in $\mathbb{E}^{**}(X)$ (see 1.2.7). The proof is in fact tautological.

Example 2.2.8. Consider the absolute oriented ring spectra of Example 1.2.2:

- (1) The first Chern class associated with the canonical orientation of motivic cohomology is nothing else than the canonical isomorphism:

$$\text{Pic}(X) \simeq \text{CH}^1(X) \simeq \mathbf{H}_M^{2,1}(X).$$

As this is an isomorphism of groups, the associated FGL is additive, *i.e.* the motivic Eilenberd-MacLane ring spectrum $\mathbf{H}_M\mathbb{Z}$ is additive.

- (2) Similarly, for the ring spectrum \mathbf{H}_ϵ represented by a mixed Weil cohomology, the first Chern class associated with the canonical orientation is induced by the cycle class:

$$c_1^\epsilon : \text{Pic}(X) \simeq \text{CH}^1(X) \rightarrow \mathbf{H}_\epsilon^2(X).$$

Therefore, the oriented ring spectrum \mathbf{H}_ϵ is additive.

- (3) For the orientation on the algebraic K-theory spectrum \mathbf{KGL}_S , it follows easily from the definition that the first Chern class of a line bundle L over a smooth S -scheme X is given by the formula:

$$c_1^{\mathbf{KGL}}(L) = \beta^{-1} \cdot (1 - [L^\vee]).$$

One deduces that the FGL associated with \mathbf{KGL}_S is the multiplicative one with parameter $-\beta$:

$$F_{\mathbf{KGL}}(x, y) = x + y - \beta \cdot xy.$$

- (4) many more examples have been constructed: an algebraic version of the Brown-Peterson spectrum ([Vez01]), of the Morava K-theory³¹, elliptic ring spectrum ([LYZR19]). These constructions are all based on the next theorem and on the motivic version of the Landweber exactness theorem [NSØ09].

2.2.c. Algebraic cobordism and the Lazard ring. We end up this course with a discussion of the motivic analog of Quillen's theorem on complex cobordism and the Lazard ring.

Let us recall that Lazard has proved that there is a universal formal group law $(\mathbb{L}, F_{\mathbb{L}})$, whose coefficient ring $\mathbb{L} = \mathbb{Z}[b_1, b_2, \dots]$ is a polynomial algebra over \mathbb{Z} with infinitely many variables. In particular, for any scheme S , the canonical orientation on \mathbf{MGL}_S gives rise to a canonical map

$$\varphi_S : \mathbb{L} \rightarrow \mathbf{MGL}^{2*,*}(S)$$

where the right hand-side is the graded part $(2n, n)$ of the algebraic cobordism ring of S . In his ICM talk in 1998, Voevodsky made the following conjecture (see [Voe98]):

Conjecture. *For any regular local scheme S , the map φ_S is an isomorphism.*

Here is the current best result on this conjecture.

Theorem 2.2.9 (Levine, Hoyois, Spitzweck). *Let S be a local pro-smooth scheme over a field of characteristic exponent e or a DVR of mixed characteristic $(0, e)$.*

³¹initially proposed by Voevodsky in a 1995 preprint

Then the map $\varphi_S[1/e]$ is an isomorphism, and the algebraic cobordism $\mathbf{MGL}_S[1/e]$ is the universal formal group law.

In fact, Levine was the first to give the proof, in [Lev09], of the above theorem when S is the spectrum of a field k of characteristic 0. It was in fact a corollary of the isomorphism between the geometric part of the cohomology represented by \mathbf{MGL}_k and the “concrete” algebraic cobordism theory, defined by explicit generators and relations, by Levine and Morel ([LM07]). Indeed, Levine and Morel had proved that the analogous of the above theorem for their cobordism theory in characteristic 0.

Hoyois ([Hoy15]) proved the above theorem directly when S is a field of characteristic exponent $e = p$, by coming back to the strategy of Quillen, using the following fundamental relation between algebraic cobordism and the motivic Eilenberg-MacLane spectrum: under the above assumption, the canonical morphism of motivic spectra

$$\mathbf{MGL}_S/(b_1, b_2, \dots) \rightarrow \mathbf{H}_M \mathbb{Z}_S$$

is an equivalence after inverting e . The global strategy as well as the above isomorphism were devised by Hopkins and Morel (unpublished).

Then Spitzweck proved the theorem in the case stated above in [Spi20], by using the preceding result, extended to a general S using his construction of the motivic Eilenberg-MacLane spectrum, and also by using the slice filtration.

EXERCICES

Exercice 1. Prove that infinite suspension of Thom spaces induce a functor on the groupoid of virtual vector bundles. Using the method of Riou: see 4.1 in <http://www.math.u-psud.fr/~riou/doc/operations.pdf>.

Eventually: explain the construction of Bachmann-Hoyois: [BH21, 16.1] ?

Exercice 2 (FGL). Prove the existence and unicity of the exponential strict isomorphism associated to a rational FGL. [Str19]

Show the existence of the Lazard ring.

Show Lazard theorem. (cf. <https://people.math.harvard.edu/~lurie/252xnotes/Lecture2.pdf>, +lecture3,4) ?

Exercice 3. Compute Chern classes of a tensor product for multiplicative FGL.

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