Conventions. Schemes will be Noetherian. Smooth will mean smooth of finite type. Unless stated otherwise, the Grothendieck topology on $\mathscr{S}m_S$ is the Nisnvich topology. Therefore, sheaves will mean sheaves for the Nisnevich topology.

We use the language of ∞-categories.[1] We let Cat_∞ (resp. $Cat_\infty^\otimes$) be the ∞-category of presentable ∞-categories with left adjoint ∞-functors (resp. presentable symmetric closed monoidal ∞-category with left adjoint and symmetric monoidal ∞-functors). All our ∞-categories will be presentable ∞-categories. Similarly, monoidal ∞-categories will be presentable monoidal ∞-categories. On the other hand, we mostly work in the associated homotopy category in this course.

Monoidal means symmetric monoidal. All our monoids are commutative. We denote by $\mathbb{1}_S$ the "sphere spectrum" over S *i.e.* the unit of the monoidal structure on $\mathrm{SH}(S)$.

Unless stated otherwise, spectrum means motivic spectrum, and ring spectrum means commutative motivic spectrum.

Contents

[1] One can obtain an explicit presentation of the categories here by using classical model categories on simplicial sheaves: the invective and Nisnevich-local model category structures on simplicial sheaves, and their Bousfield $\mathbb{A}^1$-localization.

Cours 1. Oriented spectra and Chern classes

Introduction

The theory of *characteristic classes* of *fiber bundles* arose at exactly the same time than (singular) cohomology, in 1935. That year, Stiefel (in his PhD) and Whitney both introduced the notion of fiber bundle and some associated characteristic class.[2] Meanwhile, at the Moscow international conference on topology, Alexander and Kolmogorov independently introduced cohomology and the (soon to be called) cup-product.

The history of the subject of characteristic classes was then marked by the introduction of *Pontryagin classes*, out of the computation of the homology of real grassmanianns, by Pontryagin in 1942, and by the introduction of *Chern classes*, obtained through the determination of the cohomology of complex Grassmanians by Chern in 1946. A last event I want to mention is the course "characteristic classes" given at the University of Princeton by John Milnor in 1957.[3]

Here is a list of the characteristic classes that emerged from the works mentioned above:[4]

name	fiber b.	notation	group
Stiefel-Whitney	smooth real v.b.	w_i	$H^i(B, \mathbb{Z}/2)$
Pontryagin	smooth real v.b.	p_i	$H^{4i}(B, \mathbb{Z})$
Chern	smooth complex v.b.	c_i	$H^{2i}(B, \mathbb{Z})$

Note that fiber bundle had already appeared: first in works of E. Cartan on Lie groups and their associated homogeneous spaces, and in the work of Stiefel, slightly earlier in 1933, who was interested in constructing new 3-dimensional varieties (in view of the Poincaré conjecture). Recall that in the most general form, a fiber bundle is a map $p : E \to B$ such that there exists an open cover $W \to B$ and a W-homeomorphism: $(F \times W) \to E \times_B W$ for some space F.[5] One calls

[2]The terminology "fiber bundle" is due to Stiefel, though he actually only considered smooth real vector bundles, while it was Whitney that formally introduced the so-called characteristic classes.

[3]Notes by Stasheff were available at that time. They were finally published in 1974, [MS74].

[4]Recall that Pontryagin classes are actually particular cases of Chern classes according to the formula: $p_i(V) = (-1)^i.c_{2i}(V \otimes_{\mathbb{R}} \mathbb{C})$, where $V \otimes_{\mathbb{R}} \mathbb{C}$ is the complexification of the real vector space V.

[5]Variants arise first by working in other categories than topological spaces. In algebraic geometry, one also considers covers from various (Grothendieck) topologies: Zariski, Nisnevich, étale, fppf (mainly).

B, E, F respectively the base (space), the total space and the fiber (space) of the fiber bundle.[6]

Example 1.0.1. Here are some of the most famous examples of fiber bundles:

- *tangent bundles.* $p : TM \to M$, projection from the tangent bundle of a smooth (resp. analytic) manifold M. This is a particular case of smooth real (resp. complex analytic) vector bundles.
- *homogeneous spaces.* for G a Lie group, and $H \subset G$ a closed subgroup, $p : G \to G/H$. This is a particular case of a principal G-bundle.
- *Covering spaces.* $P \to X$. The fiber is then a *discrete* space.
- The Möbius strip T is (non trivially!) fibered over S^1, the map $T \to S^1$ being the projection.
- The Hopf fibration: $S^3 \to S^2$, with fiber S^1.

In the previous list, only vector bundles were considered. In topology, more general fiber bundles naturally appear in the so-called *obstruction theory*. They arise as morphisms in the Postnikov tower, in good cases (simple, or more generally nilpotent spaces). The attendees have already seen this theory at work in the talk of Aravind Asok: primary and secondary obstructions can be seen as characteristic classes In this course however, we will focus on algebraic vector bundles, in order to draw a picture similar to the above table.

Characteristic classes are invariant under isomorphism of fiber bundles. In particular, they can differentiate the homotopy type of the total space. However, they are far from determining this homotopy type (let alone the diffeomorphism type), even if one adds the homotopy type of the base.[7] In his groundbreaking 1954 work on cobordism, Thom proved a therefore very surprising fact: the cobordism class of an unoriented closed smooth manifold M is completely determined by the so-called Pontryagin numbers, which are computed through Pontryagin classes of the tangent space of M. This was the beginning of a deep revolution in algebraic topology, which contribute to led to generalized cohomology theories, aka *spectra* such as cobordism, complex (real, Morava,...) K-theory, elliptic cohomology, and the beautiful picture painted by chromatic homotopy theory.

[6]Remark that such a fiber space is in particular a Hurewicz fibration, and therefore a Serre fibration. So one sometimes abusively says “fibration” for “fiber bundle”.

[7]This can already be seen in the classification of Seifert fibrations, mentioned above.

In this talk, we will consider the theory of characteristic classes as it was developed in motivic homotopy theory, after Voevodsky, Morel, Panin, Levine, and many more! The authors interest on the subject arose during his PhD under the supervision of Fabien Morel, during the years 1999-2002. This interest has grown during all my carrier (as can be seen in one's bibliography). I would like to seize this opportunity to thank Fabien again to having shared his visions on, and led me to, this wonderful world of motivic homotopy.

1.1. Stable motivic homotopy

1.1.a. **Motivic spectra.**

1.1.1. *Stable homotopy theory.* Recall from the preceding talks that the $\mathbb{A}^1$-homotopy category $\mathrm{H}^{\mathbb{A}^1}(S)$ over a scheme S is obtained by considering the ∞-topos $\mathrm{Sh}^\infty(\mathscr{S}m_S)$ of Nisnevich sheaves over the smooth site $\mathscr{S}m_S$ and by localizing it further with respect to $\mathbb{A}^1$-homotopy: that is we invert for any smooth S-scheme X the maps $\mathbb{A}^1_X \to X$ in $\mathrm{Sh}^\infty(\mathscr{S}m_S)$ via the Yoneda embedding.

On the associated ∞-category $\mathrm{H}^{\mathbb{A}^1}_\bullet(S)$ of pointed objects in $\mathrm{H}^{\mathbb{A}^1}(S)$, we even get a symmetric monoidal ∞-category where the tensor product is the so-called smash product.

We have seen in the talk of F. Morel that there are several models of spheres in motivic homotopy theory: the simplicial sphere S^1, the multiplicative group $(\mathbb{G}_m, 1)$, and the projective line $(\mathbb{P}^1, \infty)$. All objects are considered in $\mathrm{H}^{\mathbb{A}^1}_\bullet(S)$ without indicating the bases scheme S (which plays no specific role here) in the notation. And they are related by the relation:

$$\mathbb{P}^1 \simeq S^1 \wedge \mathbb{G}_m \tag{1.1}$$

As in classical topology, we obtain the stable motivic homotopy category by $\otimes$-inverting the third model of sphere, $\mathbb{P}^1$. For completeness, we will now state the main theorem that will give us our fundamental category (see also the talk of Kirsten Wickelgren).

Theorem 1.1.2 (Robalo)**.** *Let S be any scheme. There exists a universal presentable monoidal ∞-category $\mathrm{SH}(S)$ equipped with a monoidal ∞-functor:*

$$\Sigma^\infty : \mathrm{H}^{\mathbb{A}^1}_\bullet(S) \to \mathrm{SH}(S)$$

which admits a right adjoint Ω^∞ and such that $\Sigma^\infty \mathbb{P}^1$ is $\otimes$-invertible.[8]

[8]An object X in a monoidal ∞-category is $\otimes$-invertible if the ∞-functor $X \otimes ?$ is an equivalence of categories.

Actually, the proposition could be stated more abstractly for an arbitrary presentable monoidal ∞-category and an arbitrary object $\mathbb{S}$. Under, this form the proof is due to Robalo: [Rob15].

1.1.3. It follows from the construction, and the isomorphism (1.1) that, all the possible spheres $\mathbb{S} = \mathbb{P}^1_S, \mathbb{G}_{m,S}, S^1$, becomes $\otimes$-invertible after applying Σ^∞.

The resulting ∞-category acquires a very important property: it is stable in the sense of [Lur17, Def. 1.1.1.9]. This implies that the associated homotopy category admits a triangulated structure (see [Lur17, 1.1.2.13]). Note that the suspension functor for this triangulated structure is given by the formula:

$$\mathbb{E}[1] = \mathbb{E} \otimes \Sigma^\infty S^1.$$

Definition 1.1.4. The monoidal ∞-category $\mathrm{SH}(S)$ is called the *stable motivic homotopy category* over S. Its objects are called *motivic spectra* over S.

The unit object with respect to the monoidal structure is denoted by $\mathbb{1}_S$. One defines the *Tate twist* as $\mathbb{1}_S(1) = \Sigma^\infty \mathbb{P}^1[-2] = \Sigma^\infty \mathbb{G}_{m,S}[-1]$. By construction, this is a $\otimes$-invertible objects in $\mathrm{SH}(S)$ so that one also denotes by $?(n)$ the n-th tensor product with respect to this object.

If this does not cause confusion, we will denote by

$$[\mathbb{E}, \mathbb{F}]_S = \mathrm{Hom}_{\mathrm{SH}(S)}(\mathbb{E}, \mathbb{F}) = \pi_0 \, \mathrm{Map}_{\mathrm{SH}(S)}(\mathbb{E}, \mathbb{F})$$

the abelian group of morphisms in the homotopy category associated to the ∞-category $\mathrm{SH}(S)$. We usually even drop the index S in the notation.

It might be useful to have in mind the classical model for motivic spectra[9] given here without taking care about the monoidal structure. A model for a motivic spectrum $\mathbb{E}$ is the data of a sequence $(\mathbb{E}_n)_{n\geq 0}$ where $\mathbb{E}_n$ is a pointed simplicial Nisnevich sheaf together with suspension maps:

$$\mathbb{P}^1 \wedge \mathbb{E}_n \to \mathbb{E}_{n+1}.$$

1.1.b. **Representable cohomology theories.** For us, the main function of the stable homotopy category is that its objects, the $\mathbb{P}^1$-spectra, represent cohomology theory. The originality of the theory is that these cohomology theories are *bigraded*.

[9]That is the objects of an underlying model category whose associated ∞-category is equivalent to the above one.

Definition 1.1.5. *Cohomology theories.* Let $\mathbb{E}$ be a motivic spectrum over S. For any smooth S-scheme X and any pair of integers $(n,i) \in \mathbb{Z}^2$, one defines the $\mathbb{E}$-cohomology of X in degree n and twists i as:

$$\mathbb{E}^{n,i}(X) = [\Sigma^\infty X_+, \mathbb{E}(i)[n]].$$

These cohomologies have the distinctive features of being contravariant, additive, $\mathbb{A}^1$-homotopy invariant and $\mathbb{P}^1$-stable. Moreover, one gets long exact sequences of Mayer-Vietoris type but with respect to Nisnevich distinguished squares.

Example 1.1.6. There are many examples of cohomology theories which are representable in the stable motivic homotopy category, over a given base field $S = \mathrm{Spec}(k)$.

(1) All the classical Weil cohomologies admits canonical extensions over smooth k-schemes which are representable.[10]
   - $\mathrm{char}(k) = 0$: algebraic de Rham cohomology;
   - $\mathrm{char}(k) = p > 0$: rigid cohomology (Berthelot)
   - given an embedding $\sigma : k \subset \mathbb{C}$, the rational singular cohomology of the σ-complex points of a smooth k-scheme X; this is called simply the rational Betti cohomology.
   - give a prime ℓ invertible in k, the $\mathbb{Q}_\ell$-adic étale cohomology of $X \otimes_k \bar{k}$; this is called the geometric $\mathbb{Q}_\ell$-adic cohomology.

   We will denote by $\mathbf{H}_\epsilon$ the spectrum representing one of these Weil cohomologies: $\epsilon =$dR, rig, B, ℓ respectively. In all these cases, twists does not change the cohomology up to an isomorphism (see *loc. cit.* Introduction before theorem 1).

(2) Note that Betti cohomology can be taken with integral coefficients. It is still representable in $\mathrm{SH}(k)$, and twists do not change the cohomology (up to an isomorphism as above). We will denote by $\mathbf{H}_\sigma R$ the corresponding spectrum over k with coefficients in a ring R.

(3) Given now a real embedding $\sigma : k \subset \mathbb{R}$. One can consider the integral singular cohomology of the real points:

$$H^n(X^\sigma(\mathbb{R}), \mathbb{Z}).$$

   This is representable by a ring spectrum that will be denoted by $\mathbf{H}_\sigma \mathbb{Z}$. In that case, twists just shift cohomology degrees, again up to isomorphisms:

$$(\mathbf{H}_\sigma \mathbb{Z})^{n,i}(X) = H^{n-i}(X^\sigma(\mathbb{R}), \mathbb{Z}).$$

[10] This has been axiomatized in the notion of *mixed Weil cohomology theory* in [CD12].

Example 1.1.7. The following examples are absolute cohomology in the sense of Beilinson. In the motivic homotopy categorical sense, it means that the base scheme does not really matter when one computes the associated representable cohomology.[11]

(1) Motivic cohomology with coefficients in a ring R, $\mathbf{H}_{\mathrm{M},S}R$ can be defined for any scheme S.[12] The distinctive feature of motivic cohomology is:

$$\begin{aligned} \mathbf{H}_{\mathrm{M}}^{2n,n}(X,\mathbb{Z}) &= \mathrm{CH}^n(X) && X/S \text{ smooth, } S = \text{field, Dedekind ring} \\ \mathbf{H}_{\mathrm{M}}^{n,n}(k,\mathbb{Z}) &= \mathrm{K}_n^{\mathrm{M}}(k) && k \text{ any field.} \end{aligned}$$

where $\mathrm{CH}^n(X)$ denotes the Chow group of X: classes of codimension n algebraic cycles up for the rational equivalence, and $\mathrm{K}_n^{\mathrm{M}}(X)$ is the n-th Milnor K-group: the tensor algebra over the abelian group $k^\times$ modulo the Steinberg relation.

(2) Quillen algebraic K-theory over a regular scheme S. This is represented by a spectrum that we will denote by $\mathbf{KGL}_S$. This spectrum is periodic, in the sense that there exists a canonical isomorphism, the "Bott isomorphism":

$$\beta : \mathbf{KGL}_S(1)[2] \to \mathbf{KGL}_S.$$

Taken into account this isomorphism one gets the following distinctive property for any smooth S-scheme X:

$$\mathbf{KGL}^{n,i}(X) = K_{2i-n}(X)$$

where the right hand-side is Quillen algebraic K-theory: the $(2i-n)$-th homotopy group of the nerve of the Q-category associated with the exact category of vector bundles over X (pointed by the 0-object).

If one wants a true absolute spectrum, one will replace Quillen K-theory by Weibel homotopy invariant K-theory. We will still denote by $\mathbf{KGL}_S$ the resulting spectrum, for an arbitrary base scheme S.

[11] Concretely, this means that these cohomologies are representable by an *absolute motivic spectrum*: a collection of motivic spectra $\mathbb{E}_S$ over any scheme S, equipped for any morphism $f : T \to S$, with an isomorphism $f^*(\mathbb{E}_S) \simeq \mathbb{E}_T$ satisfying the usual cocycle condition. This is also a cartesian section of the fibred category SH over the category of schemes.

[12] The first definition of such a spectrum is due to Voevodsky. At the time being, one uses a definition based on higher Chow groups and due to Spitzweck. Both definitions coincide if S is smooth over a field.

(3) Algebraic cobordism over any base scheme S, denoted by $\mathbf{MGL}_S$. We will recall later the definition of this spectrum, and state a universal property.

1.1.8. *Extended cohomology.* Representable cohomology theory automatically acquire more structures.For once, a motivic spectrum $\mathbb{E}$ defines a family of contravariant functors:

$$\tilde{\mathbb{E}}^{n,i} : \left(\mathrm{H}_\bullet^{\mathbb{A}^1}(S)\right)^{op} \to \mathscr{A}b, \mathcal{X} \mapsto [\Sigma^\infty \mathcal{X}, \mathbb{E}(i)[n]].$$

Formally, this functor turns cofiber sequences in $\mathrm{H}_\bullet^{\mathbb{A}^1}(S)$ into long exact sequences of abelian groups.

An interesting remark is that the $\mathbb{E}$-cohomology is therefore invariant under weak motivic equivalences.[13]

Secondly, one immediately gets a definition of cohomology with support. A closed S-pair (X, Z) will be a pair of schemes such that X is a smooth S-scheme, and $Z \subset X$ a closed subscheme. By taking homotopy cofibers, in the pointed motivic homotopy category, one can define the object $X/X - Z$ which fits into a cofiber sequence:

$$(X - Z)_+ \xrightarrow{j_*} X_+ \to X/X - Z$$

One defines the $\mathbb{E}$-cohomology of X with support in Z in degree n and twist i as:

$$\mathbb{E}_Z^{n,i}(X) := \tilde{\mathbb{E}}^{n,i}(X/X - Z).$$

Therefore, it fits into a long exact sequence:

$$\dots \mathbb{E}_Z^{n,i}(X) \to \mathbb{E}^{n,i}(X) \xrightarrow{j^*} \mathbb{E}^{n,i}(X - Z) \xrightarrow{\partial_{X,Z}} \mathbb{E}_Z^{n+1,i}(X) \dots$$

This cohomology with support enjoys good properties:

(1) Contravariance: for any morphism $f : Y \to X$ of smooth S-schemes, there exists a pullback functor:

$$f^* : \mathbb{E}_Z^{n,i}(X) \to \mathbb{E}_{f^{-1}(Z)}^{n,i}(Y).$$

(2) Covariance: for any closed immersion $i : T \to Z$ of closed subschemes of X, one gets:

$$i_* : \mathbb{E}_T^{n,i}(X) \to \mathbb{E}_Z^{n,i}(X).$$

Remark 1.1.9. (1) All the previous examples admit a natural notion of cohomology with support, which agree with the above definition.

[13] To get a nice picture on weak motivic equivalences, we refer the survey paper [AOsr21].

(2) Morel-Voevodsky purity theorem implies the following property of cohomology with support, in the case where $Z \subset X$ is a smooth subscheme of codimension c:

$$\mathbb{E}_Z^{n,i}(X) \simeq \mathbb{E}^{n-2c,i-c}(Z).$$

1.1.c. **Ring spectra and cup-products.** The next definition is the last piece of structure one needs on cohomology to get characteristic classes.

Definition 1.1.10. A *(commutative) ring spectrum* $\mathbb{E}$ over the base scheme S is a (commutative) monoid object in the homotopy category associated with $\mathrm{SH}(S)$.

In particular, the structure of a ring spectrum on $\mathbb{E}$ is given by a unit $1_{\mathbb{E}} : \mathbb{1}_S \to \mathbb{E}$ and a product $\mu : \mathbb{E} \otimes_S \mathbb{E} \to \mathbb{E}$, which satisfies the usual axioms. If one wants to be precise, we will say that $(\mathbb{E}, \mu, 1_{\mathbb{E}})$ is a motivic ring spectrum.

One deduces a product on $\mathbb{E}$-cohomology, which is often called the cup-product[14]: given cohomology classes:

$$a : \Sigma^\infty X_+ \to \mathbb{E}(i)[n], b : \Sigma^\infty X_+ \to \mathbb{E}(j)[m]$$

one defines $a \cup_\mu b$ as the composite map:

$$\Sigma^\infty X_+ \xrightarrow{\delta_*} \Sigma^\infty(X \times_S X)_+ = \Sigma^\infty X_+ \otimes_S \Sigma^\infty X_+ \xrightarrow{a \otimes b} \mathbb{E} \otimes \mathbb{E}(i+j)[n+m] \xrightarrow{\mu} \mathbb{E}(i+j)[n+m].$$

We will usually denote this product simply as ab.

It follows that for any smooth S-scheme X, $\mathbb{E}^{**}(X)$ is a bi-graded algebra over the bigraded ring $\mathbb{E}^{**}(S)$, usually called the *coefficient ring* of $\mathbb{E}$ and simply denoted by $\mathbb{E}^{**}$.

Remark 1.1.11. One should be careful that the above bigraded algebra is not simply graded commutative with respect to the first index. To state the required formula one neesd the special element $\epsilon \in [\mathbb{1}_S, \mathbb{1}_S]$, which acts as a scalar on any representable cohomology theory, defined by the switch map inverse map $x \mapsto x^{-1}$ on $\mathbb{G}_m$, and using that $\Sigma^\infty(\mathbb{G}_m, 1) = \mathbb{1}_S(1)[1]$.

Then the ϵ-graded commutativity formula, for a, b as above, reads as follows:

$$ab = (-1)^{n+m-i-j}.\epsilon^{i+j}.ba \tag{1.2}$$

The proof is formal once one notices that ϵ can also be defined, up to $\mathbb{A}^1$-homotopy, by the map switching the factors on $\mathbb{G}_m \times \mathbb{G}_m$ (see [Mor04, Lemma 6.1.1]).

[14]This terminology, due to Whitney for the product on singular cohomology, has firmly remained in algebraic topology, due to the tremendous importance of its introduction in the thirties.

Example 1.1.12. All the examples of cohomology theories of Example 1.1.6 and Example 1.1.7 are in fact representable by motivic ring spectra, and the associated cup-product corresponds to their usual product.

Remark 1.1.13. The theory developed below only requires the above definition. However, all the examples considered admits a *highly structured product*, *i.e.* it is the object in the homotopy category associated with a (commutative) algebra object of the monoidal ∞-category $\mathrm{SH}(S)$. Beware that in general, it is fundamental in classical (and motivic) stable homotopy theory to give a clear distinction between those two kinds of structure.

1.1.d. **Representability of the Picard group.**

1.1.14. In algebraic topology, given an abstract group G, one can define its classifying space BG as an explicit simplicial set: the nerve of the groupoid associated with G, made of a single object $*$, a morphism for any element of g, the composition being given by the group law.

It is more common to consider topological groups (eg. Lie groups) G, and then one can still define a classifying space BG (as an explicit CW-complex) with the distinctive feature that for any topological space X, the homotopy classes of (unpointed) maps $[X, BG]$ are in bijhection with the principal homogeneous G-spaces.

In motivic homotopy theory, Morel and Voevodsky have provided an analog of the second construction but using the first one and the framework of simplicial sheaves. For an algebraic group G over a scheme S, and a smooth S-scheme X, we denote by $H^1_{\mathrm{Nis}}(X, G)$ the set of G-torsors on X for the Nisnevich topology. Let us state a particular case of Morel-Voevodsky's construction relevant in our case.

Proposition 1.1.15. *Let S be a scheme, and G be an algebraic group over S.*

Then there exists an object BG in $\mathrm{H}_{\bullet}^{\mathbb{A}^1}(S)$ and for any smooth S-scheme X, a canonical functorial application of (pointed) sets:

$$H^1_{\mathrm{Nis}}(X, G) \to [X, BG]_S^{un}. \tag{1.3}$$

Moreover, if the left hand-side is $\mathbb{A}^1$-invariant over all smooth S-schemes, this map is an isomorphism.

Note that it is very rare that the second condition holds. The only example we have in mind is that of $G = \mathbb{G}_m$ when S is regular. To sum-up in the case of $\mathbb{G}_m$, we get a canonical map:

$$\mathrm{Pic}(X) \to [X, B\mathbb{G}_{m,S}]_S^{un}$$

which is bijective whenever S is regular. Note also that the theory of Morel and Voevodsky shows that $B\mathbb{G}_{m,S}$ admits the geometric model that one expect: it is the infinite projective space $\mathbb{P}^\infty_S$, that is the infinite Grassmanian of lines in an affine space:

$$B\mathbb{G}_{m,S} = \varinjlim_{n\geq 0} \mathbb{P}^n_S$$

where the colimit can be taken in the category of simplicial sheaves on $\mathscr{S}m_S$ (to get an explicit model). Note that $\mathbb{P}^\infty_S$ will be seen as a pointed sheaves via the point at ∞ of all the $\mathbb{P}^n_S$. We will recall from this discussion the canonical map:

$$\mathrm{Pic}(X) \to [X, \mathbb{P}^\infty_S]^{un}_S. \tag{1.4}$$

Remark 1.1.16. (1) Assume $S = \mathrm{Spec}(k)$ is the spectrum of a field. If one restricts our attention to smooth affine k-schemes X, then the map (1.3) is an isomorphism for an isotropic reductive k-group schemes: e.g. $G = \mathrm{GL}_n, \mathrm{SL}_n, \mathrm{Sp}_{2n}$. This is a theorem which was first obtained by Morel in certain cases, and in general by Asok, Hoyois and Wendt (see [AHW20] for the extra condition needed for G).

(2) Morel and Voevodsky also give geometric models for over classifying spaces: as an example, for any $n \geq 0$, BGL_n is equivalent to the infinite grassmanian of sub-n-vector bundles:

$$\mathrm{BGL}_n = \varinjlim_{r\geq 0} \mathrm{Gr}(n, n+r).$$

1.2. Oriented ring spectra

1.2.a. **Definition and examples.** Given the notation of the previous section, we have all the ingredients to formulate the notion of orientation, which has been introduced in motivic homotopy theory, by analogy with topology, at the time of the first proof of the Milnor conjecture by Voevodsky.

Definition 1.2.1. Let $(\mathbb{E}, \mu, 1_\mathbb{E})$ be a ring spectrum over S . Let $i : \mathbb{P}^1_S \to \mathbb{P}^\infty_S$ be the canonical inclusion of pointed Nisnevich sheaves, both being pointed by the point at ∞.

An *orientation* of $\mathbb{E}$ is the data of a class $c \in \tilde{\mathbb{E}}^{2,1}(\mathbb{P}^\infty_S)$ suth that $i^*(c) = 1_\mathbb{E}$ via the identification: $\tilde{\mathbb{E}}^{2,1}(\mathbb{P}^1_S) = \mathbb{E}^{0,0}(S)$.

We will say that the pair $(\mathbb{E}, c)$ is an oriented (ring) spectrum.

Note that an orientation can be seen as a map:

$$c : \Sigma^\infty \mathbb{P}^\infty_S \to \mathbb{E}(1)[2].$$

Example 1.2.2. (1) Let X be a smooth $\mathbb{Z}$-scheme. We have seen that there is an isomorphism: $\mathbf{H}^{2,1}_{\mathrm{M}}(X) \simeq \mathrm{CH}^1(X) \simeq \mathrm{Pic}(X)$. This extends to ind-smooth $\mathbb{Z}$-schemes. But $\mathrm{Pic}(\mathbb{P}^\infty_S) \simeq \mathbb{Z}.c$, the free abelian group generated by c, the class of the tautological invertible bundle $\lambda = \mathcal{O}_{\mathbb{P}^\infty_S}(-1)$.

Moreover, the restriction of c to $\mathbb{P}^1_S$ is the cycle class of the point at ∞. It is the unit of the ring structure on $\mathrm{CH}^*(\mathbb{P}^1_S) \simeq \mathbf{H}^{2*,*}_{\mathrm{M}}(\mathbb{P}^1_S) \simeq \mathbb{Z}[c]$.[15] Therefore, the class c corresponds to an orientation of $\mathbf{H}_{\mathrm{M}}\mathbb{Z}$ over the base scheme $\mathbb{Z}$.

Given now any scheme X, we can look at the canonical map $f : S \to \mathbb{Z}$. Then f^*, being compatible with products on motivic cohomology, $f^*(c)$ is an orientation of $f^*\mathbf{H}_{\mathrm{M}}\mathbb{Z} = \mathbf{H}_{\mathrm{M}}\mathbb{Z}_S$.

(2) Let $\mathbf{H}_\epsilon$ be the ring spectrum representing one of the mixed Weil cohomology theories, over smooth k-schemes with coefficient in the appropriate field K of characteristic 0 as in Example 1.1.6. The corresponding cohomology admits a cycle class map:

$$\mathrm{CH}^i(X) \simeq \mathbf{H}^{2i,i}_{\mathrm{M}}(X) \to \mathbf{H}^{2i,i}_{\epsilon}(X) \overset{(*)}{\simeq} \mathbf{H}^{2i,0}_{\epsilon}(X) =: \mathbf{H}^{2i}_{\epsilon}(X)$$

which is compatible with products (mapping intersection products to "cup-products").[16] Therefore, the image of $c \in \mathrm{CH}^1(\mathbb{P}^\infty_k)$ in $\mathbf{H}^{2,1}_\epsilon(\mathbb{P}^\infty_S)$ induces a canonical orientation of the mixed Weil spectrum $\mathbf{H}_\epsilon$.

(3) The same strategy works for the singular cohomology of complex points of smooth algebraic k-schemes, over $k \subset \mathbb{C}$.

(4) On the contrary, the ring spectrum representing singular cohomology of the real points of smooth algebraic k-scheme, $k \subset \mathbb{R}$, is not orientable. Indeed:

$$\mathbf{H}^{2,1}_\sigma(\mathbb{P}^1_k) = H^1(\mathbb{RP}^1, \mathbb{Z}) = \mathbb{Z}$$
$$\mathbf{H}^{2,1}_\sigma(\mathbb{P}^\infty_k) = H^1(\mathbb{RP}^\infty, \mathbb{Z}) = \mathbb{Z}/2.$$

(5) The spectrum $\mathbf{KGL}$ representing algebraic K-theory is oriented by the following class:

$$c^{\mathbf{KGL}}(L) = \beta^{-1}(1 - [\lambda^\vee]) \in \mathbf{KGL}^{2,1}(\mathbb{P}^\infty_S)$$

---

[15] This follows for example from the definition via pullback along the diagonal.

[16] We put the last isomorphism to recall that mixed Weil cohomologies are $(0,1)$-periodic. This isomorphism is non-canonical and depends on the choice of a generator of the 1-dimensional K-vector space $\mathbf{H}^{1,1}_\epsilon(\mathbb{G}_{m,k})$. See [CD12] as indicated in Example 1.1.6.

where we denote by $[L] \in \mathbf{KGL}^{0,0}(X) \simeq K_0(X)$ the class of a line bundle L/X in the Grothendieck group of vector bundles over X, and $\beta \in \mathbf{KGL}^{-2,-1}(S)$ is the Bott element (over S).

(6) The algebraic cobordism spectrum $\mathbf{MGL}_S$ admits, by construction, a canonical orientation. That will be cleared out in the next course.

1.2.b. **Chern classes.**

1.2.3. *First Chern class.* Let $(\mathbb{E}, c)$ be an oriented ring spectrum over S. Taken into account the canonical map (1.4), we obtain for any smooth S-scheme X a canonical map:

$$\begin{aligned}\mathrm{Pic}(X) \to [X_+, \mathbb{P}^\infty_S]_{\mathrm{H}^{\mathbb{A}^1}_\bullet(S)} &\xrightarrow{\Sigma^\infty} [\Sigma^\infty X_+, \Sigma^\infty \mathbb{P}^\infty_S]_{\mathrm{SH}(S)} \\ &\xrightarrow{c_*} [\Sigma^\infty X_+, \mathbb{E}(1)[2]]_{\mathrm{SH}(S)} = \mathbb{E}^{2,1}(X).\end{aligned}$$

This is called the first Chern class associated with the orientation c, denoted simply by c_1. It is clearly contravariantly functorial in the scheme X. However we must observe at this point that c_1 is simply an application, and not necessarily a morphism of groups. In fact, all the maps in the above compositum are morphisms of groups *except* the suspension map Σ^∞. This fact is extremely meaningful in the theory of oriented ring spectra (see the next course).

The key fact of the theory is the following *projective bundle formula*:

Theorem 1.2.4. *Consider the above notation. Let $V \to X$ be a rank n vector bundle over a smooth S-scheme X, and let $P = \mathbb{P}(V)$ be the associated projective bundle. We let $p : P \to X$ be the canonical projection, and let λ_P be the canonical line bundle on P (coming from the fact $\mathbb{P}(V)$ classifies sub-line bundles of V).*[17] *Then the following map:*

$$\begin{aligned}\oplus_{i=0}^{d-1}\mathbb{E}^{**}(X) &\to \mathbb{E}^{**}(P) \\ \lambda_i &\mapsto \sum_i p^*(\lambda_i).c_1(\lambda_P)^i\end{aligned}$$

is an isomorphism of $\mathbb{E}^{**}(X)$-modules.

One can reformulate the above theorem by saying that $\mathbb{E}^{**}(P)$ is a bigraded $\mathbb{E}^{**}(X)$-algebra (through the pullback map p^*) which is free of rank n, generated by $c_1(\lambda_P)^i$ for $0 \leq i \leq r-1$.

[17] This line bundle is often denoted by $\mathcal{O}_P(-1)$, for example by Fulton in [Ful98].

Remark 1.2.5. *Milnor sequence.* In general, for a ring spectrum $(\mathbb{E}, \mu, 1_{\mathbb{E}})$ over S, one always has the so-called Milnor exact sequence:

$$0 \to \varprojlim_{n\geq 0}{}^1 \mathbb{E}^{2,1}(\mathbb{P}^n_S) \to \mathbb{E}^{2,1}(\mathbb{P}^\infty_S) \to \varprojlim_{n\geq 0} \mathbb{E}^{2,1}(\mathbb{P}^n_S) \to 0.$$

It follows from the above that, whenever $\mathbb{E}$ is oriented, the left hand side vanishes as the involved inductive system satisfies the Mittag-Leffler condition. In particular, to give an orientaition on $\mathbb{E}$, it is sufficient to give classes $c_n \in \mathbb{E}^{2,1}(\mathbb{P}^n_S)$ for all $n > 0$ such that $c_1 = 1_{\mathbb{E}}$ and $\iota_n^*(c_{n+1}) = c_n$.

As a corollary, one gets our first family of characteristic classes, the Chern classes of algebraic vector bundles, following a method of Grothendieck.

Definition 1.2.6. Let $(\mathbb{E}, c)$ be an oriented (motivic) ring spectrum over S. Let X be a smooth S-scheme and V/X be a vector bundle or rank n. Then there exists a unique family $(c_i(V))_{0\leq i\leq n}$ such that the following relation holds in $\mathbb{E}^{2,1}(\mathbb{P}(V))$:

$$\sum_{i=0}^{n} p^*\big(c_i(V)\big).\big(-c_1(\lambda_P)\big)^{n-i}$$

Note in particular that $c_i(V) \in \mathbb{E}^{2i,i}(X)$. If $i > n$, we put $c_i(V) = 0$.

1.2.7. According to the above definition, we get the following properties of Chern classes:

1. *Invariance under isomorphism.* For any isomorphism $V \simeq V'$ of vector bundles over X, $c_i(V) = c_i(V')$.
2. *Compatibility with pullbacks.* For any vector bundle V/X, and any morphism $f : Y \to X$ of smooth S-schemes, $f^*c_i(V) = c_i(f^{-1}V)$.
3. *Triviality.* For a trivializable vector bundle V, $c_i(V) = 0$ if $i > 0$.
4. *Nilpotence.* Here it is important that S is noetherian. For any vector bundle V/X, and any $i \geq 0$, the Chern class $c_i(V)$ is nilpotent.

The third relation follows from the fact $c_1(\mathcal{O}_{\mathbb{P}^n}(-1))^{n+1} = 0$ (see the proof of the projective bundle theorem). The last relation is left as an exercice to the reader.

To go further, one needs the so-called *splitting principle*. It is based on the following "splitting construction".

Proposition 1.2.8. *Let X be a smooth S-scheme, and V a vector bundle over X. Then there exists a smooth projective map $p : X' \to X$*

such that $p^{-1}(E)$ splits as a direct sum of line bundles and such that for any oriented ring spectrum $\mathbb{E}$ over S, the pullback map $p^* : E^{**}(X) \to \mathbb{E}^{**}(X')$ is injective.

Remark 1.2.9. A canonical construction for X' is to take the flag bundle associated with V, which is the moduli space which parametrize the complete flag of sub-vector bundles of V. The fact the projection map induces an injective pullback on an oriented cohomology theory can be seen as a *motivic Leray-Hirsch theorem*. The latter can be obtained directly from the homotopy Leray spectral sequence of [ADN20] associated with p and with coefficients in $\mathbb{E}$.

1.2.10. *Splitting principle.* As a corollary of the previous proposition, one obtains the so-called splitting principle for Chern classes associated with any oriented ring spectrum $(\mathbb{E}, c)$ as above. Let V/X be a rank n vector bundle over a smooth S-scheme X.

First, we define the total Chern class as the polynomial in t, with coefficients in the (bigraded) ring $\mathbb{E}^{**}(X)$:[18]

$$c_t(V) = \sum_{i \geq 0} c_i(V).t^i.$$

Then the splitting principle tells us that, to compute with the Chern classes of V, one can assume that V is split using the preceding splitting construction. this amounts to say that the total Chern class splits: it admits Chern roots α_i such that:

$$c_t(V) = \prod_{i=1}^{n} (1 + \alpha_i.t)$$

Then any symmetric polynomial in the Chern roots α_i admits an expression in terms of the Chern classes of V.

As an example, one can get the formula:

Proposition 1.2.11 (Whitney sum formula)**.** *For any exact sequence of vector bundles over a smooth S-scheme X:*

$$0 \to V' \to V \to V'' \to 0$$

one has: $c_t(V) = c_t(V').c_t(V'')$.

Example 1.2.12. Consider the above notation. Given V/X a vector bundle of rank n, one usually defines the *Euler class* of V as:

$$e(V) = c_n(V).$$

[18] This convention for total Chern class follows Fulton [Ful98]. Other conventions, such as for example in [MS74] simply considers the sum $c(V) = \sum_i c_i(V)$ in the "total" cohomology $\bigoplus_i \mathbb{E}^{2i,i}(X)$.

Assume X is smooth affine of dimension n over $S = \mathrm{Spec}(k)$, the spectrum of an algebraically closed field (or a field in which (-1) is a sum of squares). Then we have seen in the talk of Aravind Asok that the vanishing of the Euler class in motivic cohomology, is equivalent to the fact V splits-off a trivial summand. (One direction obviously follows from the above Whitney sum formula!)

However, to remove the assumption on k, one needs a finer version of the Euler class, with values in the Chow-Witt group.

1.2.c. **The algebraic Hopf map.**

1.2.13. The endomorphism ring of the sphere spectrum, $\mathrm{End}(\mathbb{1}_S)$ acts on any motivic spectrum $\mathbb{E}$. Similarly, any map $\varphi : \mathbb{1}_S \to \mathbb{1}_S(i)[n]$ induces a morphism $\varphi \otimes \mathbb{E} : \mathbb{E} \to \mathbb{E}(i)[n]$. This can be seen as an action of the graded ring $\Pi^{n,i}(S)$ — the stable motivic cohomotopy of S — on $\mathbb{E}$.

According to the fundamental theorem of Morel, when S is the spectrum of a field k, one gets $\Pi^{n,n}(k) \simeq \mathrm{K}_n^{\mathrm{MW}}(k)$, the Milnor-Witt ring of k. Other any base S, one still gets important endomorphisms:

(1) *Algebraic Hopf map.* $\eta : \mathbb{1}_S(1)[1] \to \mathbb{1}_S$, which is induced by the canonical map $\mathbb{A}^2_S - \{0\} \to \mathbb{P}^1_S, (x, y) \mapsto [x : y]$ (in coordinates).
(2) *Classes of units.* for any $u \in \mathcal{O}(S)^\times$, one deduces $[u] : \mathbb{1}_S \to \mathbb{1}_S(1)[1]$ from the map $u : S \to \mathbb{G}_{m,S}$ corresponding to u. One then puts:

$$< u >= 1 + \eta.[u],$$

which is an element in degree $(0, 0)$ of the bigraded ring $\Pi^{**}(S)$.

Note that one can check that $\epsilon = - < -1 > \in \Pi^{0,0}(S)$, where ϵ was defined in Remark 1.1.11.[19]

As a consequence of the projective bundle theorem and using the above mentioned remark, one deduces:

Proposition 1.2.14. *Let $\mathbb{E}$ be an orientable ring spectrum. Then the algebraic Hopf map η acts trivially on $\mathbb{E}$: $\eta \otimes \mathbb{E} = 0$.*

As a consequence, for every units $u \in \mathcal{O}(S)^\times$, $< u >$ acts by the identity. In particular, ϵ acts by (-1): $\epsilon \otimes \mathbb{E} = -\mathrm{Id}_{\mathbb{E}}$. As a consequence, relation (1.2) *becomes*

$$ab = (-1)^{nm}.ba.$$

[19] In fact, η and ϵ are defined by pullbacks from elements of $\Pi^{**}(\mathbb{Z})$. It is likely that $\Pi^{0,0}(\mathbb{Z}) = \mathbb{Z}[\epsilon]/(\epsilon^2 = 1)$. This would be a direct consequence of the absolute purity property for (reduced) closed subschemes of $\mathrm{Spec}(\mathbb{Z})$.

Proof. The first assertion follows from the cofiber sequence in the pointed motivic homotopy category:

$$\mathbb{A}^2 - \{0\} \xrightarrow{\eta} \mathbb{P}^1_S \xrightarrow{\iota_1} \mathbb{P}^2_S$$

for which we refer to [Mor04]. Indeed, if $\mathbb{E}$ is oriented, then $\mathbb{E} \otimes \iota_1$ is a split monomorphism. The rest of the assertions follow easily. □

In general, the action of the Hopf map is not sufficient to detect orientability of a ring spectrum. However, we have the notable theorems.

Theorem 1.2.15. *Let k be a perfect field, and $\mathbb{E} \in \mathrm{SH}(k)$ be a homotopy module with a ring structure. Then the following conditions are equivalent:*

(i) *$\mathbb{E}$ is orientable.*
(ii) *$\eta \otimes \mathbb{E} = 0$.*
(iii) *$\mathbb{E}$ admits transfers in the sense of Voevodsky (*i.e. *action of finite correspondences).*

This theorem uses the equivalence between homotopy modules with transfers and Rost cycle modules: see [Dég13]. We can now obtain a more direct proof by using the equivalence of homotopy modules with Milnor-Witt cycle modules: see [Fel21].

Theorem 1.2.16 (Morel, Cisinski-D.)**.** *Let $\mathbb{E}$ be a rational motivic ring spectrum over a scheme S. Then the following conditions are equivalent:*

(i) *$\mathbb{E}$ is orientable.*
(ii) *$\eta \otimes \mathbb{E} = 0$.*
(iii) *$\epsilon \otimes \mathbb{E} = -\mathrm{Id}_{\mathbb{E}}$.*

In fact in these case, $\mathbb{E}$ is a rational motive !

Sketch of proof.[20] The proof relies on Morel's decomposition of the rational stable homotopy category into:

$$\mathrm{SH}(S)_{\mathbb{Q}} \simeq \mathrm{SH}(S)_{\mathbb{Q}+} \times \mathrm{SH}(S)_{\mathbb{Q}-}$$

characterized by the equivalent properties:

(i) $\mathbb{E} \in \mathrm{SH}(S)_{\mathbb{Q}+}$ (resp. $\mathbb{E} \in \mathrm{SH}(S)_{\mathbb{Q}-}$).
(ii) $\epsilon \otimes \mathbb{E}$ is equal to -1 (resp. $+1$).
(iii) $\eta \otimes \mathbb{E}$ is null (resp. invertible).

Then the main point is to show that the canonical map:

$$\mathbb{1}_S \otimes \mathbb{Q}_+ \to \mathbf{H}_{\mathrm{B},S}$$

[20] This proof is a simplification of the proof given in [CD19, Th. 16.2.13].

is an isomorphism, where the right hand-side is Beilnson motivic cohomology ring spectrum (representing the 0-th graded piece of rational algebraic K-theory, over regular schemes). By a localization arguments and invariance under inseparable field extensions, one reduces to the case of a perfect field k. Then a devissage argument (we use rational coefficients at this point) reduces to the preceding theorem.

Remark 1.2.17. As a complement, let us say that one now knows how to compute both the plus and the minus part of rational motivic stable homotopy category (see [DFJK21]):

$$\mathrm{SH}(S)_{\mathbb{Q}+} \simeq \mathrm{DM}(S, \mathbb{Q})$$

is the category of rational mixed motivic complexes. In particular, rationally, being orientable is the same as being a motivic complex.

For the minus part, one has:

$$\mathrm{SH}(S)_{\mathbb{Q}-} \simeq \mathbf{H}\,\underline{\mathrm{W}}_{S \otimes_{\mathbb{Z}} \mathbb{Q}} - \mathrm{mod}$$

where the right hand-side is the category of modules over the unramified rational Witt sheaf, seen over the caracteristic 0 part $S \otimes_{\mathbb{Z}} \mathbb{Q}$ of S (in particular, it is zero on a scheme of positive charadteristic).

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