16.07.2024

Problems

PCMI GSS 2024 - $\mathbb{A}^1$-algebraic topology (following F. Morel)

1 Show that the presheaf $X \mapsto \mathcal{O}^*(X)$ is $\mathbb{A}^1$-invariant.

2 Show that $K_{Nis}(\mathcal{O}^*, 1)$, the Nisnevich sheafification of $K(\mathcal{O}^*, 1)$ is $\mathbb{A}^1$-invariant. Note: $K(G, n)$ denotes the n-th Eilenberg Maclane space.

3 Show that $X \mapsto \mathcal{O}(X)$ is not $\mathbb{A}^1$-invariant. What is $L_{mot}\mathcal{O}$?

4 A presheaf of sets F is said to be birational if $F(X) = F(U)$ for every dense open immersion $U \hookrightarrow X$, and if $F(\sqcup_i X_i) = \prod_i F(X_i)$. Show that such an F is a Nisnevich sheaf.
If, furthermore, F is a presheaf of abelian groups, show that F is strictly $\mathbb{A}^1$-invariant. Deduce that for a smooth projective curve C of genus ≥ 1, the free additive presheaf $\mathbb{Z}[C]$ is strictly $\mathbb{A}^1$-invariant.

5 Let k be a field. A k-scheme X is said to be $\mathbb{A}^1$-rigid if for any $Y \in Sm_k$, the projection map $\mathbb{A}^1 \times Y \to Y$ induces a bijection $X(Y) \to X(Y \times \mathbb{A}^1)$. Show that the following schemes are $\mathbb{A}^1$-rigid:

(a) $\mathbb{G}_m$.

(b) A curve of genus ≥ 1.

Compute $\pi_0^{\mathbb{A}^1}$ for the above schemes.