IAS Summer Collaborators Report

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1. Motivation and Description of the Problem

Our group is interested in the wave turbulence phenomenon in the dispersive models. In these models, the Hamiltonian structures typically allow for the control of certain low Sobolev norms (energy) of solutions. However, numerical experiments have shown that energy can transfer from low to high frequencies, even though the total energy remains bounded. As a result, solutions can become increasingly oscillatory (chaotic) over time. Mathematically, this provides a framework for modeling wave turbulence in dispersive equations. Typically, such energy transfer can be captured by the growth of high Sobolev norms (H^s norm with large regularity s).

Since the pioneering works of Bourgain in the 90's, research has focused on:

- Constructing solutions that have their H^s norms actually go to infinity with certain rates. (Usually referred to as "lower bounds of H^s norm growth".)
- Finding a good control of the possible H^s norm growth. (Usually referred to as "upper bounds of H^s norm growth".)

After discussion, we have decided to attack the problems by first studying the "upper bounds", which are more closely related to our previous research experience, particularly in global well-posedness and (modified) scattering in dispersive models. Among many dispersive equations, we are particularly interested in an one-dimensional model introduced by Majda, McLaughlin, and Tabak (MMT) in 1997 [MMT97]

$$i\partial_t u = (-\Delta)^\alpha u \pm D^{-\beta}\left[\left|D^{-\beta}u\right|^2 (D^{-\beta}u)\right]. \tag{1}$$

Here, $u = u(t, x)$ is a function of time $t \in \mathbb{R}$ and one space variable x on the torus $\mathbb{T}$ or real line $\mathbb{R}$, and $D = -i\nabla_x$. There are two parameters in (1): $\alpha \in (1/2, 1)$ and $\beta \geq 0$.

As presented, the MMT model is a two-parameter family of one-dimensional dispersive equations designed to assess the validity of weak turbulence theory for random waves in an unambiguous and transparent manner. Among these two parameters, α determines the dispersion relation and offers the potential to cover a broader range of Schrödinger equations with fractional Laplacian operators (potentially leading to weaker dispersion), while β introduces a smoothing effect in the nonlinearity (resulting in a smoother nonlinearity). These two modifications to the equation and the chemical interplay between them enable the exploration of wave turbulence phenomena from different perspectives.

This model is quite rich in the sense that when $\alpha = 1$ and $\beta = 0$, (1) becomes the usual Schrödinger equation with cubic nonlinearity:

$$i\partial_t u = -\Delta u \pm |u|^2 u. \tag{2}$$

When $\alpha = 1/4$, its linear part mimics the water wave dispersion law.

During the two-week visit to IAS, we focused on understanding the long-time behavior of (1) on both the compact manifold setting ($x \in \mathbb{T}$) and the real line setting ($x \in \mathbb{R}$). In the following, we discuss these two cases separately, as quite different behaviors are expected.

- When equation (1) is considered on $\mathbb{T}$, we expect that the nonlinear term will dominate, and the aforementioned low-to-high energy transfer will occur. We are interested in obtaining a good upper bound for the high Sobolev norms of u, i.e., $\|u(t,\cdot)\|_{H^s}$ with large s, as $t \to \infty$.
- When equation (1) is considered on $\mathbb{R}$, we expect the (linear) dispersion term $(-\Delta)^\alpha u$ to dominate. Specifically, we expect solutions to (1) to scatter to linear solutions $e^{it(-\Delta)^\alpha} u_+$ as $t \to \infty$. In particular, this implies that high Sobolev norms of u do not grow.

2. Progress and Future Plan

Next, we summarize our progress during these two weeks and outline our research plan following the IAS visit.

On the torus, $\mathbb{T}$. In history, there have been efforts to obtain upper bounds on the growth of high Sobolev norms in dispersive models. For (1), the best upper bound so far is a polynomial growth bound in [EGT19]:

$$\|u\|_{H^s} \leq C(1+t)^{\frac{s-\alpha}{2\alpha-1}} \tag{3}$$

when $\alpha \in (1/2, 1]$ and $\beta = 0$. In [Thi17], Thirouin obtained a weaker upper bound via a different method. As a starting point, during our time at the IAS, we adapted the methods from [EGT19] to (1) with nonzero β. We obtain the following result:

Theorem 1. *Let u be a solution to (1) with an a priori control of the H^α norm. Then, for large enough s,*

$$\|u(t,\cdot)\|_{H^s} \leq C(1+t)^{\frac{s-\alpha+\beta}{2\alpha-1+\beta}}. \tag{4}$$

Note that (4) recovers (3) when $\beta = 0$, and demonstrates a better rate when $\beta > 0$. When (1) is defocusing (i.e., takes a "+" sign), the H^α norm of u is always bounded, due to the conservation of its Hamiltonian $\int(\frac{1}{2}|(-\Delta)^{\alpha/2}u|^2 + \frac{1}{4}|D^{-\beta}u|^4)\,dx$. However, in the focusing ("$-$") case, we must impose such an *a priori* bound, at least to ensure the solution exists for all time.

Based on Theorem 1, we plan to proceed. The growth rate in (4) is expected to be not sharp: Results in [CKO12, Theorem 1.2] suggest that at least when $\beta = 0$, one should expect an upper bound with the exponent converging to $4(s-1)/9 + \varepsilon$ as α approaches 1. However, the exponent in (4), when $\beta = 0$, converges to $s - 1$ as $\alpha \to 1$. To close this gap, we plan to better exploit the high-low frequency interactions in the nonlinear term of (1), which are origins of the energy transfer behaviors.

From our heuristic computations, it appears that the presence of more high frequency components could slow down the energy transfer, leading to a slower growth of Sobolev norms. To this end, one idea to sharpen the above result is to combine the techniques from Bourgain [Bou93], which essentially introduces a time-dependent high-low frequency cutoff, with those from [EGT19]. Another approach is to combine the normal forms method, which we currently used to derive Theorem 1, with the upside-down I method as in [CKO12], and work more delicately with the resonant system of the model. This approach was successfully employed in [CKO12] for nonlinear Schrödinger equations, where the authors improved the Sobolev norm growth rates obtained earlier in [Sta97] and [Soh11].

We believe that these two directions would give better results, and we have conducted some preliminary calculations along these lines. We will continue our computation after the visit.

On the real line, $\mathbb{R}$. In the context of Euclidean space, one would anticipate more favorable long-term behavior in comparison to compact manifolds, specifically in terms of scattering. In this context, scattering refers to the phenomenon where a nonlinear solution eventually resembles a linear solution within the same model. Our focus has also extended to examining the MMT model on the real line. During the visit, we performed some preliminary calculations and believed that scattering should occur for (1) with $\alpha < 1$ and $\beta \geq 0$. In contrast to the case on $\mathbb{T}$, this implies that the Sobolev norms of solutions do not exhibit growth over time. To our knowledge, there is no existing scattering result for (1), even when $\beta = 0$.

In many scattering analyses of dispersive models, it is commonly known that Morawetz-type estimates play an important role. These estimates refer to certain space-time controls of nonlinear solutions to dispersive models, which can be viewed as nearly conserved quantities. However, deriving such Morawetz-type estimates is usually not simple. In our computation, there are at least two major difficulties when deriving Morawetz-type estimates for (1):

- Low spatial dimension and the "non-symmetry" introduced by $(-\Delta)^\alpha$. In [BHL16], similar estimates are derived for nonlinear Schrödinger equations, i.e., (1) with $\alpha < 1$ and $\beta = 0$, but on higher (three or more) spatial dimensions. To derive a version for lower spatial dimensions (in our case, one dimension), one typically attempts to prove an "interactive" version. However, unlike the usual Schrödinger case ($\alpha = 1$), it is not easy to adapt the methods in [BHL16] to one dimension, partly due to certain non-symmetry issues introduced by $(-\Delta)^\alpha$.
- Non-power type nonlinear term. When establishing a Morawetz-type inequality, it is important for the nonlinearity to exhibit a specific algebraic structure, in order to make a full time derivative of some format of the solution. For instance, traditional power-type nonlinearities have been extensively employed in previous works [CKS$^+$10, PV09]. However, in the MMT setting, the distinct structure of the nonlinear term presents challenges in realizing a complete derivative from the nonlinearity.

We are actively working to address these challenges. One direction we have been exploring is to approximate (1) with equations those have "better" nonlinear structures. We attempt to use frequency localization for (1) and employ different approximations at different frequencies. If successful, this approach should develop a method for treating similar equations with non-power type nonlinearities.

Another direction we have been exploring is to first prove some weaker estimates rather than Morawetz-type ones, which are still related to dispersion-like behaviors. One possibility is to study the decay-in-time of L^2-norms within any large ball. Demonstrating such an estimate could indicate that a portion of a solution's mass escapes to far-end, reflecting a characteristic dispersion-like behavior. In this direction, some results were obtained in [Tao04]. This is a new approach, as there are no prior results that prove scattering without a Morawetz-type estimate, and we plan to further study this in the future.

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