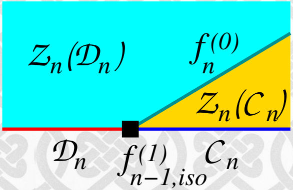
Generalized symmetry – a bird-eye/holographic perspective

Xiao-Gang Wen (MIT)

(holo-eq) Symmetry in n-dim space $\leftrightarrow$ Braided fusion n-category

2024/07, PiTP

Kong Wen 1405.5858
Kong Wen Zheng 1502.01690
Ji Wen 1905.13279
Ji Wen 1912.13492
Kong Lan Wen Zhang Zheng 2005.14178



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Generalized global symmetries

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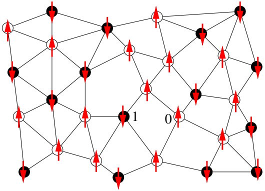
Top-down view = bird-eye view = viewing from one higher dimension

Symmetry (quantum, internal, old picture)



- A **local quantum system** is described by $(\mathcal{V}_N, H)$. (1) $\mathcal{V}_N$: a Hilbert space with a tensor decomposition $\mathcal{V}_N = \otimes_{i=1}^N \mathcal{V}_i$. (2) H: a local Hamiltonian acting on $\mathcal{V}_N$, $H = \sum \hat{O}_{ij}$
- A **symmetry** for a local quantum system:

  (1) $\{U_g \mid g \in G,\ U_g U_h = U_{gh},\ U_g^\dagger = U_{g^{-1}}\}$, where U_g act on the Hilbert space $\mathcal{V} = \otimes_i \mathcal{V}_i$.

  (2) The set of **local symmetric operators**: $\{O_i^{\text{symm}} \mid U_g O_i^{\text{symm}} = O_i^{\text{symm}} U_g\}$.

  (3) The Hamiltonian $H = \sum O_i^{\text{symm}}$.

  (4) The allowed symmetric perturbations: $\delta H = \sum \delta O_i^{\text{symm}}$

Symmetry (quantum, internal, new picture)

- A **symmetry** for a local quantum system = **a choice of subset of local operators**, which form a **vector space with product** $\mathcal{A} = \{O = \sum_\alpha O_\alpha \mid aO + bO' = cO'',\ O_\alpha O_\beta = f^\gamma_{\alpha\beta} O_\gamma\}$.

- $\{O_\alpha\}$ is a basis of $\mathcal{A}$ formed by local operators.

- We refer to $\mathcal{A}$ as a **local-operator sub-algebra**.

- Symmetric Hamiltonian $H = \sum O_i,\ O_i \in \mathcal{A}$.

- The algebra of all local operators

$$\mathcal{A}_{\text{no-symm}} = \{O_\alpha \mid \text{all local operators}\}$$

corresponds to the case with no symmetry.

Holo-equivalence of symmetries

- Two symmetries $\mathcal{A}$ and $\tilde{\mathcal{A}}$ are **holo-equivalent** $\mathcal{A} \sim \tilde{\mathcal{A}}$

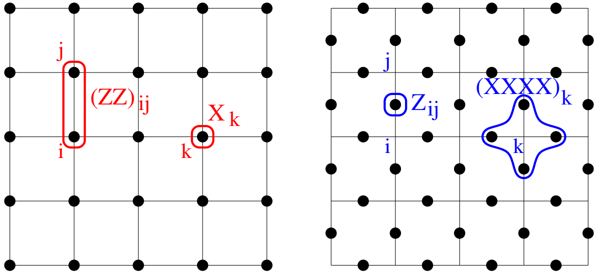
- if $\mathcal{A}$ and $\tilde{\mathcal{A}}$ are isomorphic (*ie* the local operators in $\mathcal{A}$ and $\tilde{\mathcal{A}}$ have an one-to-one correspondence $O_\alpha \leftrightarrow \tilde{O}_\alpha$, such that $f^\gamma_{\alpha\beta} = \tilde{f}^\gamma_{\alpha\beta}$, where $O_\alpha O_\beta = f^\gamma_{\alpha\beta} O_\gamma$ and $\tilde{O}_\alpha \tilde{O}_\beta = f^\gamma_{\alpha\beta} \tilde{O}_\gamma$.

- or if $\mathcal{A} \otimes \mathcal{A}_{\text{no-symm}} \sim \tilde{\mathcal{A}} \otimes \tilde{\mathcal{A}}_{\text{no-symm}}$

Ji Wen 1912.13492; Kong Lan Wen Zhang Zheng 2005.14178; Chatterjee Wen 2205.06244

- This leads to a notion of **Holo-equivalent classes of symmetries, called categorical symmetries $\leftrightarrow$ Isomorphic classes of local-operator sub-algebra (up to $\otimes\mathcal{A}_{\text{no-symm}}$)**

Two examples



- Spin-$\frac{1}{2}$ on sites of square lattice, with local-operator sub-algebra

$$\mathcal{A}_{\text{site}} = \{Z_i Z_{i+x},\ Z_i Z_{i+y},\ X_k \mid \text{generators}\}$$

- Spin-$\frac{1}{2}$ on links of square lattice, with local-operator sub-algebra

$$\mathcal{A}_{\text{link}} = \{Z_{i,i+x},\ Z_{i,i+y},\ X_{i,i+x}X_{i,i-x}X_{i,i+y}X_{i,i-y} \mid \text{generators}\}$$

- The two local operator algebras are isomorphic and correspond to holo-equivalent symmetries.

Symmetry (looking into holo-equivalence class)

- The symmetry transformations for the **categorical symmetry** (defined by the local-operator sub-algebra $\mathcal{A}$):

$$\mathcal{A}_{\text{trans}} = \{O_{\text{trans}} \mid O_{\text{trans}} O_\alpha = O_\alpha O_{\text{trans}},\ \ O_\alpha \in \mathcal{A}\}$$

- $\mathcal{A}_{\text{trans}}$ is an operator algebra describing symmetry transformation.
- $\mathcal{A}_{\text{trans}}$ depends on the homotopy types of space: $\mathcal{A}_{\text{trans}} = \mathcal{A}_{\text{trans},0}^{\otimes \pi_0(M)}$.

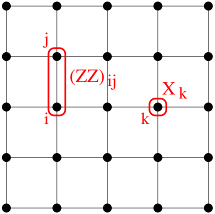
- Isomorphic algebras of the transformation operators classify/define symmetries. Isomorphic local-operator sub-algebras may have non-isomorphic transformation algebra. **A holo-equivalent class of symmetries may contain different symmetries**.
- If we can find a set of generators $\{U_g\}$ of $\mathcal{A}_{\text{trans}}$ that satisfy a group-like algebra $U_g U_h = U_{gh}$, then the symmetry is a invertible group-like symmetry.
- If such set of generators does not exist, then the symmetry is non-invertible → **Algebraic (higher) symmetry**

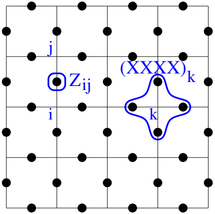
Our two examples: $\mathbb{Z}_2^{(0)} \xlongequal{\text{holo-equivalent}} \mathbb{Z}_2^{(1)}$ in 2+1D

- The two local-operator sub-algebras → same **categroical symm**

$$\mathcal{A}_{\text{site}} = \{Z_i Z_{i+x}, \quad Z_i Z_{i+y}, \quad X_k \}$$
$$\mathcal{A}_{\text{link}} = \{Z_{i,i+x}, \quad Z_{i,i+y}, \quad X_{i,i+x}X_{i,i-x}X_{i,i+y}X_{i,i-y} \}$$

- But they correspond to different symmetries, $\mathbb{Z}_2^{(0)}$ 0-form symmetry and $\mathbb{Z}_2^{(1)}$ 1-form symmetry:
$\mathcal{A}^{\text{site}}_{\text{trans}} = \{\prod_i X_i\} \to \mathbb{Z}_2^{(0)}$.
$\mathcal{A}^{\text{link}}_{\text{trans}} = \{\prod_{\langle ij\rangle \in \text{loop}} Z_{ij}\} \to \mathbb{Z}_2^{(1)}$.





- 2+1D $\mathbb{Z}_2^{(0)}$ 0-form and $\mathbb{Z}_2^{(1)}$ 1-form symmetries are holo-equivalent.
The systems with the two symmetries have the same phase diagram, the same set of critical points, the same local low energy dynamics. **Holo-equivalent symmetries can be mapped into each other via duality transformations (gauging)**

Nussinov Ortiz cond-mat/0605316
Gaiotto Kapustin Seiberg Willett 1412,5148

More examples holo-equivalent symmetries

- In 1+1D, $\mathbb{Z}_2 \times \mathbb{Z}_2$ with the mixed anomaly is holo-equivalent to $\mathbb{Z}_4$ symmetry. Chatterjee Wen 2203.03596; Zhang Levin 2206.01222

- In 1+1D, $\mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2$ with a triple-mixed anomaly is holo-equivalent to D_4 symmetry (*ie* $\mathcal{D}^\omega(\mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2) = \mathcal{D}(D_4)$).
- In 1+1D, $\mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2$ with another triple-mixed anomaly is holo-equivalent to Q_8 symmetry. Goff Mason Ng math/0603191

- In 1+1D, G symmetry is holo-equivalent to $\mathcal{R}\mathrm{ep}(G)$ symmetry, which is non-invertible if G is non-Abelian.

An example of non-invertible $(n-1)$-symmetry

- A n-dimensional lattice model with states on link-ij labeled by $g_{ij} \in G$. The algebra of symmetry transformations is given by

$$\mathcal{A}_{\text{trans}} = \{U_q|\{g_{ij}\}\rangle = \mathrm{Tr} \prod_{ij\in\text{loop}} R_q(g_{ij})|\{g_{ij}\}\rangle \mid q \text{ is a rep of } G\}$$

$$U_{q_1}U_{q_2} = \sum_{q_3} N^{q_3}_{q_1,q_2} U_{q_3}, \quad q_1 \otimes q_2 = \oplus_{q_3} N^{q_3}_{q_1,q_2} q_3$$

$$\mathcal{A}_{\text{localops}} = \{f(g_{ij}), \quad |g_{ij}h_j^{-1}, h_j g_{jk}\rangle\langle g_{ij}, g_{jk}|\}\}$$ Ji Wen 1912.13492

Kong Lan Wen Zhang Zheng 2005.14178; Bhardwaj Schafer-Nameki Wu 2208.05973

- Such a $(n-1)$-symmetry (denoted as $n\mathcal{R}\text{ep}(G)$) is a non-invertible higher symmetry (*ie* an algebraic higher symmetry) if G is non-Abelian. This $(n-1)$-symm is holo-equivalent to G 0-symm.
- More examples of intrinsic non-invertible symmetry come from duality maps of self-dual models, Choi Cordova Hsin Lam Shao 2111.01139; Kaidi Ohmori Zheng 2111.01141; Bhardwaj Bottini Schafer-Nameki Tiwari 2204.0656; ... Seiberg Shao 2307.02534; Chatterjee Aksoy Wen 2405.05331; Gorantla *etal* 2406.12978 as well as from local-operator sub-algebras. Chatterjee Ji Wen 2212.14432

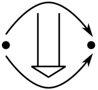
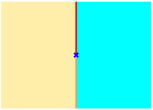
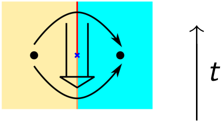
Two claims (unified theory for generalized symm)

- **Claim 1**: **Holo-equivalent classes of (generalized) symmetries in n-dimensional space are classified by braided fusion n-categories $\mathcal{M}$ in trivial Witt-class (*ie* the center of a fusion n-category.)** Kong Wen Zheng 1502.01690; Ji Wen 1912.13492 Kong Lan Wen Zhang Zheng 2005.14178
- **Claim 2**: **Anomaly-free (generalized) symmetries in n-dimensional space are classified by local fusion n-categories.** Kong Lan Wen Zhang Zheng 2005.14178
  Both symmetry charges and symmetry defects are described by (potentially different) local fusion n-categories, $\mathcal{R}_{\text{char}}$ and $\mathcal{R}_{\text{def}}$.

  (Generalized) symmetries are classified by fusion n-categories $\mathcal{R}_{\text{def}}$, which describes symmetry defects.
  Thorngren Wang 1912.02817 (1+1D); Freed Moore Teleman 2209.07471
- Two symmetries $\mathcal{R}_{\text{def}}$ and $\mathcal{R}'_{\text{def}}$ are holo-equivalent if their **center** is the same $\mathcal{Z}(\mathcal{R}_{\text{def}}) = \mathcal{Z}(\mathcal{R}'_{\text{def}})$

What is a fusion higher category?

Fusion 2-category | Extended excitations | Their relation

2+1D spacetime with world-sheet, world-line, and instanton

- (1) Co-dimension-1 excitation (objects of category) (2) Co-dimension-2 excitation = interface between co-dimension-1 excitations, ... → **Fusion *n*-category**
  (Co-dimension-n = 0-dimension in spacetime, *ie* instantons)

- Symmetry charges in n-dimensional space form a fusion higher category $\mathcal{R}_{\text{char}}$.

- Symmetry defects in n-dimensional space form a dual fusion higher category $\tilde{\mathcal{R}}_{\text{def}}$.

What is a braided fusion higher category?

Braided fusion 2-category | Extended excitations | Their relation

3+1D spacetime with world-sheet, world-line, and instanton

- (1) Co-dimension-2 excitation (objects of category) (2) Co-dimension-3 excitation = interface between co-dimension-2 excitations, ... → **Braided fusion n-category $\mathcal{M}$**
  (Co-dimension-n = 0-dimension in spacetime, *ie* instantons)

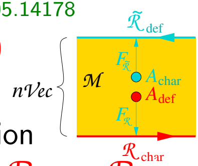
- $\mathcal{M}$ describes the (extended) excitations in a topological order in $(n+1)$-dimensional space (also denoted by $\mathcal{M}$). The excitations on a gapped boundary are described by a fusion n-category $\mathcal{R}$. The boundary $\mathcal{R}$ uniquely determine the bulk $\mathcal{M}$: $\mathcal{M} = \mathcal{Z}(\mathcal{R})$ ($\mathcal{Z}$ is called a center functor) → **topological holographic principle**.

Kong Wen 1405.5858; Kong Wen Zheng 1502.01690

Local fusion n-category classify anomaly-free symm

- A symmetry $\tilde{\mathcal{R}}_{\text{def}}$ is **anomaly-free** if there exists a symmetric system with a gapped non-degenerate ground state on any closed spaces. Thorngren Wang 1912.02817 (1+1D); Kong Lan Wen Zhang Zheng 2005.14178
- A fusion n-category $\mathcal{R}$ is **local** if there exists another fusion n-category $\tilde{\mathcal{R}}$ (dual of $\mathcal{R}$) with the same center, such that the tensor product of $\mathcal{R}$ and $\tilde{\mathcal{R}}$ over their center $\mathcal{M} = \mathcal{Z}(\mathcal{R}) = \mathcal{Z}(\tilde{\mathcal{R}})$ is trivial: $\mathcal{R} \otimes_{\mathcal{M}} \tilde{\mathcal{R}} = n\mathcal{V}\text{ec}$. Kong Lan Wen Zhang Zheng 2005.14178

- **Physical picture**: the stacking of the dual-pair $(\mathcal{R}, \tilde{\mathcal{R}})$ through their bulk $\mathcal{M}$ is a trivial topological order $\rightarrow$



- A local fusion n-category $\mathcal{R}$ always has a dual local fusion n-category $\tilde{\mathcal{R}}$. An anomaly-free symmetry described by $\mathcal{R}_{\text{def}} = \mathcal{R}$ always has a dual anomaly-free symmetry described by $\mathcal{R}_{\text{def}} = \tilde{\mathcal{R}}$.

- Excitations in local fusion n-category have integral quantum dims
  Proof: Boundary excitation $\mathcal{R}$ can be viewed as $n\mathcal{V}\text{ec}$-excitations

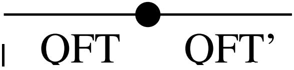
Homomorphism between QFTs

How to understand/derive the above two claims?

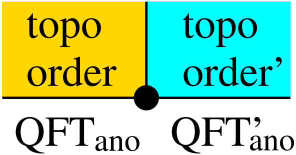
Homomorphism, as a map that preserve some structure, is the most important concept in mathematics.

What is the Homomorphism between quantum field theories?

- A morphism between two anomaly-free quantum field theories is given by a domain wall



- A morphism between two anomalous quantum field theories is given by a domain wall on the boundary and a domain wall in the bulk topological order that describes the anomaly



\- **Topological holographic principle**: the gravitational anomaly (obstruction to have lattice UV completion, usually non-invertible) in a theory QFT_{ano} is given by its **center** $\mathcal{M} = \mathcal{Z}(QFT_{ano})$, *ie* by its bulk topological order $\mathcal{M}$.

Kong Wen 1405.5858

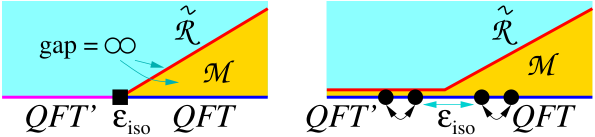
Homomorphism between QFTs with grav anomaly

- If we only interested in local low energy properties, *ie* long distance correlations of local operators, we can define two quantum field theories, QFT and QFT', to be **local-low-energy equivalent** if

$$QFT \boxtimes Gapped = QFT' \boxtimes Gapped'.$$

Kong Wen Zheng 1502.01690

- Local-low-energy homomorphism between anomalous QFT and QFT': QFT' is exact simulated by the composite theory of $QFT, \mathcal{M}, \tilde{\mathcal{R}}\ \varepsilon_{\text{iso}}:\ QFT' \cong QFT \boxtimes_{\mathcal{M}} \tilde{\mathcal{R}}$, which is called **an isomorphic holographic decomposition**.



- Local low energy excitations of QFT can be embedded into QFT' $\to$ Local-low-energy homomorphism $(\epsilon, \tilde{R}) : QFT \to QFT'$
- If the $n+1$D QFT's are gapped, they become fusion n-categories, and $(\tilde{\mathcal{R}}, \varepsilon_{\text{iso}})$ is the monoidal functor between fusion n-categories.

Symmetry $\cong$ non-invertible gravitational anomaly

- A symmetry is generated by an unitary operator U. H commutes with U: $UH = HU$.
- Now, we consider a symmetric system restricted in the symmetric sub-Hilbert space

$$U\mathcal{V}_{\text{symmetric}} = \mathcal{V}_{\text{symmetric}}.$$

The symmetry U acts trivially within $\mathcal{V}_{\text{symmetric}}$.

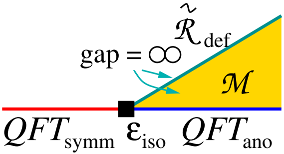
How to know there is a symmetry? How to identify the symmetry?

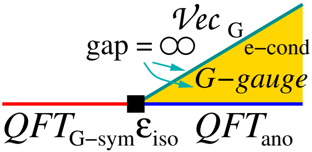
- The total Hilbert space $\mathcal{V}_{\text{tot}}$ has a tensor product decomposition $\mathcal{V}_{\text{tot}} = \otimes_i \mathcal{V}_i$, where i labels sites, because the system live on lattice.
- The symmetric sub-Hilbert space $\mathcal{V}_{\text{symmetric}}$ does not have a tensor product decomposition $\mathcal{V}_{\text{symmetric}} \neq \otimes_i \mathcal{V}_i$, indicating the presence of a symmetry.
- Lack of tensor product decomposition $\to$ **gravitational anomaly** $\to$ **Symmetry $\cong$ Non-invertible gravitational anomaly $\cong$ Topological order in one higher dimension**

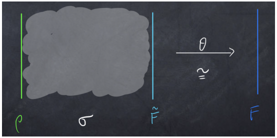
Ji Wen 1912.13492; Kong Lan Wen Zhang Zheng 2005.14178

Isomorphic holographic decomposition→Symmetry

The **isomorphic holographic decomposition** expose the hidden symmetry (*ie* the hidden gravitational anomaly) in QFT_{symm}







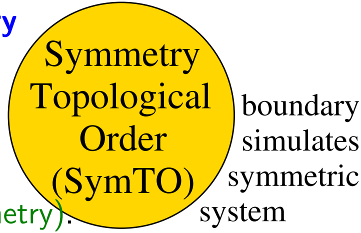
- The bulk TO $\mathcal{M}$ is the **symmetry-topological order** (SymTO). **SymTOs classify generalized symmetries in one lower dim (up to holo-equivalence).** The top boundary excitations are gapped, which form a fusion category $\tilde{\mathcal{R}}_{\text{def}}$ of symmetry defects. Since $\mathcal{M} = \mathcal{Z}(\tilde{\mathcal{R}}_{\text{def}})$, **the $n+1$D (anomaly-free) generalized symmetries are classified by (local) fusion n-category $\tilde{\mathcal{R}}_{\text{def}}$**.

Fröhlich Fuchs Runkel Schweigert 0909.5013 (1+1D); Kong Lan Wen Zhang Zheng 2005.14178

- A generalized symmetry is described by a pair (quiche *"keesh"*) $(\rho, \sigma) = (\tilde{\mathcal{R}}_{\text{def}}, \mathcal{M} = \mathcal{Z}(\tilde{\mathcal{R}}_{\text{def}}))$. Freed Moore Teleman 2209.07471

Symmetry ≅ topological order in one higher dim

- **Systems with a (generalized) symmetry (restricted within $\mathcal{V}_{\text{symmetric}}$) can be fully and exactly simulated by boundaries of a topological order**, called **SymTO** (has lattice UV completion and no symmetry) or **SymTFT** (has symmetry).



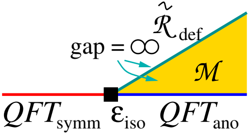
Ji Wen 1912.13492; Kong Lan Wen Zhang Zheng 2005.14178

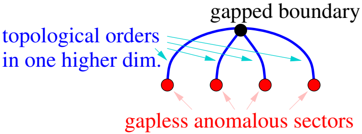
Apruzzi Bonetti Etxebarria Hosseini Schafer-Nameki 2112.02092

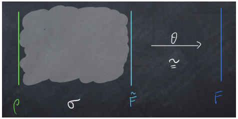
→ **Symm/TO correspondence**

- SymTO or SymTFT was originally called **categorical symmetry** in Ji Wen 1912.13492; Kong etal 2005.14178

- Three pictures of SymTO: Taco, Spider, Sandwich







Kong Wen Zheng 1502.01690; Ji Wen 1912.09391; Freed Moore Teleman 2209.07471

Operator algebra → (braided) fusion n-categories

From a local-operator sub-algebra $\mathcal{A}$ (that defines a symmetry), how to discover the braided fusion n-category, that describes the SymTO/SymTFT?

- **Braided fusion n-category $\mathcal{M}$ for a symmetry**:

- The local operators in the sub-algebra $\mathcal{A}$ carries zero total symmetry charge/defect, but can be viewed as hopping of symmetry charge/defect (quantum current).

- We use the local operators to construct a MPO patch operator with **transparent** bulk (transparent = invisible, topological defect):

$$O_{\text{patch}} = \sum \prod_{i\in\text{patch}} O_i, \;\; O_{\text{patch}} O_i = O_i O_{\text{patch}}, \quad O_i \in \mathcal{A}, \;\; i \text{ far from } \partial\text{patch}$$

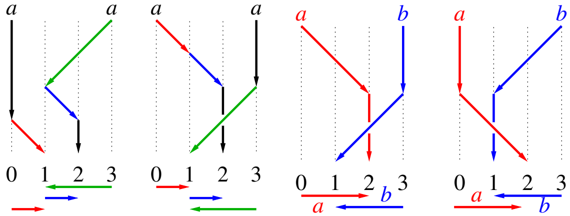
The boundary of such patch operator is symmetry charge/defect.

Ji Wen 1912.13492; Chatterjee Wen 2205.06244; Lan Zhou 2305.12917

Example: For 1+1D $\mathbb{Z}_2$ symmetry, $\mathcal{A} = \{Z_i Z_{i+1}, X_i\}$, transparent patch operators $O^{\text{char}}_{i,j} = \prod_{i\le k<j} Z_k Z_{k+1} = Z_i Z_j$, $O^{\text{def}}_{i,j} = \prod_{i<k<j} X_k$

How to calculte braided fusion catgeory data?

- Using the transparent string operators $O^a_{i,j}$ in 1d

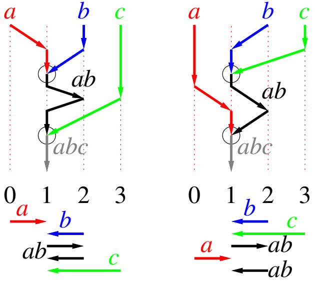


- Fusion rule:
  $O^a_{i,j} O^b_{i,j} \sim \sum_c N^{ab}_c O^c_{i,j}$
  up to local operators
  $O_i, O_j \in \mathcal{A}$.

- Self-statistics: $O^a_{10} O^a_{21} O^a_{31} = e^{i\theta_a} O^a_{31} O^a_{21} O^a_{10}$ Levin-Wen cond-mat/0302460
- Mutual-statistics: $O^b_{13} O^a_{20} = e^{i\theta_{ab}} O^a_{20} O^b_{13}$

- The F-symbol:
  $O^c_{13} O^b_{12} O^{ab}_{21} O^{ab}_{12} O^a_{10}$
  $= F^{abc} O^b_{12} O^c_{13} O^{ab}_{21} O^a_{10} O^{ab}_{12}$

Kawagoe Levin 1910.11353



- The data $(N^{ab}_c, \theta_a, \theta_{ab}, F^{abc})$ give rise to the braided fusion category $\mathcal{M}$ that describes the (generalized) symmetry.

Classify 1+1D symmetries (up to holo-equivalence)

Not every topological order describes a generalized symmetry.
Finite symmetries (up to holo-equivalence) are one-to-one classified by SymTOs in one higher dimension (*ie* Witt-trival MTCs)

- We can use 2+1D SymTOs (instead of groups) to classify 1+1D finite (generalized) symmetries (up to holo-equivalence):

# of symm charges/defects	1	2	3	4	5	6	7	8	9	10	11	12
# of 2+1D TOs	1	4	12	18	10	50	28	64	81	76	44	$\geq$212
# of symm classes (symm-TOs)	1	0	0	3	0	0	0	6	6	3	0	$\geq$3
# of (anomalous) group-symm	$1_{\mathbb{Z}_1}$	0	0	$2_{\mathbb{Z}_2^\omega}$	0	0	0	$6_{S_3^\omega}$	$3_{\mathbb{Z}_3^\omega}$	0	0	0

Ng Rowell Wen 2308.09670

- The three SymTOs at rank-12 come from Haagerup-Izumi modular data. Evans Gannon 1006.1326

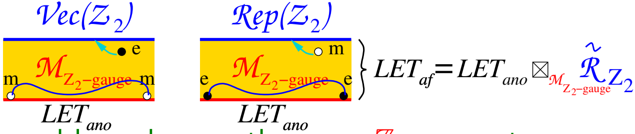
Rank 4 generalized symmetries & lattice realization

Three holo-equivalence classes of $1+1$D symmetries at rank-4:

- $\mathbb{Z}_2$**-symmetry**: SymTO $=$ $\mathbb{Z}_2$-gauge theory with anyons $\mathbf{1}, e, m, f$

- Local operator subalgebra $\mathcal{A}$ is generated by X_i, Z_iZ_{i+1}.
- Transformation algebra $\mathcal{A}_{\text{trans}}$ is generated by $U = \prod_i X_i$.

Two condensations
$A_e = \mathbf{1} \oplus e$ and
$A_m = \mathbf{1} \oplus m$ give



rise to the "same" gapped boundary $\to$ the same $\mathbb{Z}_2$ symmetry.

- When two condensations give rise to the same symmetry (same gapped boundary $\tilde{\mathcal{R}}_{\text{def}}$) $\to$ There can be a duality symmetry.

- **Anomalous** $\mathbb{Z}_2$**-symmetry**: SymTO $=$ Double-semion TO

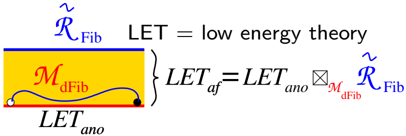
- Local-operator sub-algebra $\mathcal{A}$ is generated by Z_iZ_{i+1}, $X_i - Z_{i-1}X_iZ_{i+1}$, $Z_{i-1}X_i + X_iZ_{i+1}$.
- Transformation algebra $\mathcal{A}_{\text{trans}}$ is generated by a $\mathbb{Z}_2$ **non-on-site** transformation: $U = \prod_i X_i \prod_i \mathrm{i}^{\frac{-Z_i+Z_{i+1}+Z_iZ_{i+1}-1}{2}}$

Rank 4 generalized symmetries & lattice realization

- **Fibonacci symmetry**: SymTO = double-Fibonacci TO

Chatterjee Ji Wen 2212.14432

LET = low energy theory



Realization:

- A slab of Levin-Wen model realizing the double-Fibonacci topological order $\mathcal{M}_{\text{dFib}}$ with large gap.
- A boundary $\tilde{\mathcal{R}}_{\text{Fib}}$ with large gap from $\phi\bar{\phi}$ condensation.
- Another small-gap/gapless boundary LET_{ano}.

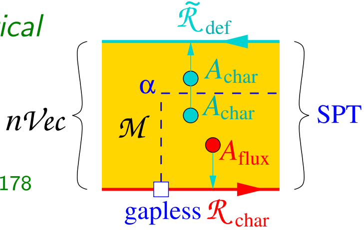
- Local operator subalgebra is generated by the operators on the low energy boundary LET_{ano}.

- Transformation algebra $\mathcal{A}_{\text{trans}}$ is generated by closed-string operators creating anyon pair (ϕ, ϕ) or $(\bar{\phi}, \bar{\phi})$, that do not condense on the $\tilde{\mathcal{R}}_{\text{Fib}}$ boundary. Symm charge $= \phi\bar{\phi}$.
  Fibonacci symm is anomalous (ϕ is not bosonic) and non-invertible (ϕ is a non-Abelian)

Wen 1812.02517

SPT phases for anomaly-free non-invertible symm

What are the symmetry protected topological (SPT) phases for an anomaly-free (non-invertible) symmetry described by fusion n-category $\tilde{\mathcal{R}}_{\text{def}}$ for its symmetry defects? Kong Lan Wen Zhang Zheng 2005.14178

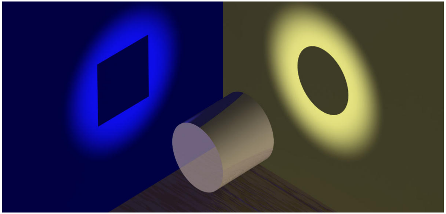


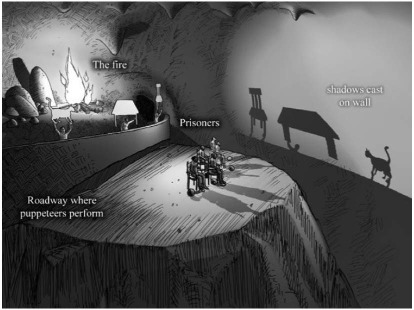
- The SPT phases for an anomaly-free (non-invertible) symmetry $\tilde{\mathcal{R}}_{\text{def}}$ is assified by the automorphisms α of the SymTO $\mathcal{M} = \mathcal{Z}(\tilde{\mathcal{R}}_{\text{def}})$, that leaves A_{char} invariant. Here A_{char} is the condensable algebra that product the gapped boundary $\tilde{\mathcal{R}}_{\text{def}}$.

- The automorphisms α of $\mathcal{M}$ are invertible domain walls in $\mathcal{M}$.
- In order for the automorphisms α to leave A_{char} invariant, the domain walls are formed by condensing symmetry charges in A_{char}.
- In order for the domain walls to be invertible, the condensation cannot be a SSB state, which is not invertible. The condensation produces a SPT phase on the domain wall.

The essence of a symmetry

- Emergent finite symmetries can go beyond groups, higher groups, and/or anomalies. But theiy can always be described by **a gappable-boundary topological order in one higher dimension** (with lattice UV completion) = **SymTO**





- The same topological order (in one higher dimensions) can have different shadows → **holo-equivalent symmetries**.

BF n-Category ↔ Generalized symmetry
(in $n + 1$D, up to holo-equivalence)