Torsors over affine curves

Problem sessions

Exercise 1. Let $d \geq 1$ be an integer and let R' be a $\mathbb{Z}/d\mathbb{Z}$–Galois extension, i.e., $\operatorname{Spec} R'$ is a $\mathbb{Z}/d\mathbb{Z}$–torsor over $\operatorname{Spec} R$. We denote by σ the canonical generator of $\mathbb{Z}/d\mathbb{Z}$.

1. Show that the formula $N(y) = y\,\sigma(y)\cdots\sigma^{r-1}(y)$ defines a group scheme homomorphism $N : R_{R'/R}(\mathbb{G}_m) \to \mathbb{G}_m$.
2. Show that $1 \to \ker(N) \to R_{R'/R}(\mathbb{G}_m) \to \mathbb{G}_m \to 1$ is an exact sequence of R–group schemes.
3. Deduce an exact sequence involving $H^1(R, \ker(N))$.
4. Show that the flat quotient of $R_{R'/R}(\mathbb{G}_m)$ by $\mathbb{G}_m$ exists in the category of schemes and is isomorphic to $\ker(N)$.
5. Construct an exact sequence
$$R^\times \to (R')^\times \xrightarrow{\sigma-1} \ker(N)(R) \to \ker\left(\operatorname{Pic}(R) \to \operatorname{Pic}(R')\right).$$
6. Discuss the case of the coordinate ring $A = R[\ker(N)]$ of $\ker(N)$.
7. For $R = \mathbb{R}$ and $S = \mathbb{C}$, is the $\mathbb{G}_m$–torsor $R_{S/R}(\mathbb{G}_m) \to R_{S/R}(\mathbb{G}_m)/\mathbb{G}_m$ trivial?

Exercise 2. Let k be a field of caracteristic p. We denote by F the Frobenius morphism.

Let G be a smooth algebraic group. Show that the map $g \mapsto g.F(g^{-1})$ is an étale isogeny, i.e., it is surjective and finite étale. Deduce that $\mathrm{SL}_{n,\overline{\mathbb{F}_p}}$ is not simply-connected in the sense of [Gro71].

Exercise 3. Let R' be a finite locally free R–algebra. Let $r \geq 0$ be an integer. Let f denote the map from $\operatorname{Spec} R'$ to $\operatorname{Spec} R$.

1. Show that the R–functor $S \mapsto \operatorname{End}_{S\otimes_R R'}\left((S\otimes_R R')^r\right)^*$ is representable by an affine R–group scheme. We denote it by $\widetilde{G} = R_{R'/R}(\mathrm{GL}_r)$ (the Weil restriction).
2. Show that a $\mathrm{GL}_{r,R'}$–torsor is locally trivialised by an open of the form $f^{-1}(U)$ where U is an open of $\operatorname{Spec} R$.
3. Show that the category of $\widetilde{G}$–torsors is equivalent to the category of locally free R'–modules of rank r.
4. Give an interpretation of the map $H^1(R, \mathrm{GL}_r) \to H^1(R, \widetilde{G})$ and show that this map is not in general injective nor surjective.

Exercise 4. Let $\mathbb{H}$ be the quartenion of Hamilton. Show that $A = \mathbb{H}[x, y]$ admits an invertible (right) $\mathbb{H}$–module which is not free.

[Hint : One can consider the exact sequence
$$0 \to P \to A^2 \xrightarrow{f} A \to 0$$
where $f : (\gamma, \mu) \mapsto (X + i)\gamma - (Y + j)\mu$.

Then one can find two solutions (γ_1, μ_1) and (γ_2, μ_2) of degree two and deduce a contradiction.]

References

[Gro71] Alexander Grothendieck. *Revêtements étales et groupe fondamental (SGA 1)*. Vol. 224. Lecture notes in mathematics. Springer-Verlag, 1971.