Characteristic classes in stable motivic homotopy theory: TA 4

Symplectic Grassmanian

Let (V, φ) be a trivial symplectic bundle of rank $2n+2$. The symplectic group $\mathrm{Sp}_{2n+2} = \mathrm{Sp}(V, \varphi)$ acts on the grassmanian $\mathrm{Gr}(2, V)$. Show that it has a natural closed orbit, whose complementary parametrizing the planes on which φ is nondegenderate, which we call the quaternionic projective space $\mathrm{H}\mathbb{P}p^n$. Show that $\mathrm{H}\mathbb{P}^1 \simeq (\mathbb{P}^1)^{\wedge 2}$ in $\mathrm{SH}(S)$.(Here $S = \mathrm{Spec}\ k$).

Sp-orientation of a Gl-orientation.

Let $(\mathbb{E}, c)$ be a GL-oriented motivic spectra. Compute the FTL associated to the Sp-orientation induced by the map $\mathrm{MSp} \to \mathrm{MGL}$. You may begin by proving that $\mathrm{b}_i(V, \varphi) = (-1)^i c_{2i}(V)$.

Chow-Witt groups.

Let (K, v) be a discretely valued field (one may assume that $K = k(C)$ is the function field of a smooth projective curve). Let $\omega_v = (\mathfrak{m}/\mathfrak{m}^2)^\vee$ be the normal cone. Show that there exists residue maps

$$\partial_v \colon \mathrm{K}_n^{\mathrm{MW}}(K) \to \mathrm{K}_{n-1}^{\mathrm{MW}}(\kappa_v, \omega_v)$$

such that ∂_v commutes with η and for $u_i \in K^\times$ and π an uniformiser with $u_1 = v_1 \pi^m$ and $v(u_i) = 0$ for $i > 1$ we have

$$\partial_v([u_1, \ldots, u_n]) = m_\varepsilon \langle \overline{v_1} \rangle [\overline{u_2}, \ldots, \overline{u_n}] \otimes \overline{\pi}^*$$

Show that the divisor map is well defined, and use this to define Chow-Witt groups. Prove that they have localisation.