Characteristic classes in stable motivic homotopy theory: TA 3

Thom spaces on virtual vector bundles.

Prove that infinite suspension of Thom spaces induces a functor

$$\mathrm{Th}\colon \mathcal{V}(X) \to \mathrm{HoSH}(S)$$

on the 1-group completion $\mathcal{V}(X)$ of the groupoid of vector bundles (you do not have to check all the details). Explain to the TA the ∞-categorical construction of Bachmann and Hoyois in [BH21, Section 16.1].

Strongly dualizable objects of SH.

For any smooth S-morphism $p : X \to S$, and any virtual bundle v over X, we put $\Pi_S(X, v) = f_*(\mathrm{Th}(v) \otimes f^!(\mathbb{1}_S))$.

1. Compute $\Pi_S(X, 0)$ for X/S smooth.
2. Show that for X/S smooth and proper the object $\Pi_S(X, 0)$ is strongly dualisable with dual $\Pi_S(X, -\mathrm{T}_X)$.
3. Let $Z \subset X$ be a closed subscheme, and X/S smooth and proper. Show that if Z is smooth over S, the object $\Pi_S(X \setminus Z, 0)$ is dualisable. Extend this result for normal crossing S-schemes.
4. Compute the dual in the previous example.
5. Give an example of a non strongly dualisable object in $\mathrm{SH}(S)$. Can you give a non dualisable object of $\mathrm{SH}(S)^\omega$?

Riemman-Roch for curves.

Make the Grothendieck-Riemman-Roch formula explicit in the case of the Chern character to get the "usual" formula for smooth projective curves over a field. Deduce facts with it.

References

[BH21] Bachmann, T. & Hoyois, M. Norms in motivic homotopy theory. *Astérisque.*, ix+207 (2021), https://doi.org/10.24033/ast