1 Characteristic classes in stable motivic homotopy theory: TA 2

The Lazard ring.

1. Show the existence of the Lazard ring $\mathbb{L}$, that corepresent the functor FGL on rings that sends a ring R to $\mathrm{FGL}(R)$ the set of two variables formal group laws with coefficients in R.

2. Assume that R is a $\mathbb{Q}$-algebra and let $F \in R[[x, y]]$ be a formal group law. Show that there exists an unique formal series $f \in R[[t]]$ with $f(0) = 0$ and $f'(0) = 1$ such that $f(x + y) = F(x, y)$. It is the exponetial $\exp_F$ of F.

   We will show that the Lazard ring is a polynomial ring.

3. Show that $\mathbb{L}$ comes naturally with a grading, such that the coefficients $c_{i,j}$ of the universal formal group law $F = \sum_{i,j} c_{i,j} x^i y^j$ are of degree $i + j - 1$.

4. Construct a natural graded ring homomorphism

   $$\varphi \colon \mathbb{L} \to \mathbb{Z}[b_1, b_2, \dots, b_n, \dots].$$

   (Hint: twist the addidive formal group law by the universal formal power series.)

   Denote by $I \subset \mathbb{L}$ the ideal generated by elements of positive degrees, and by $J = (b_i)_{i \geqslant 1} \subset \mathbb{Z}[b_1, b_2, \dots]$.

   We admit the following **fact** (see [Lur]): For every integer $n > 0$, the ring homomorphism φ induces an injection $(I/I^2)_n \simeq (J/J^2)_n \simeq \mathbb{Z}$, in a way that the image is $p\mathbb{Z}$ if $n + 1 = p^m$ for some m, and $\mathbb{Z}$ otherwise.

5. Construct a graded surjective morphism

   $$\psi \colon \mathbb{Z}[t_1, \dots, t_n, \dots] \to \mathbb{L}.$$

   Show that it is an isomorphism.

Computations of Chern classes of tensor products.

In the case of the additive and the multiplicative formal group laws, compute the total Chern classes of a tensor product.

References

[Lur] Jacob Lurie. Lecture 3 on Lazard's theorem. Available at https://people.math.harvard.edu/~lurie/252xnotes/Lecture3.pdf.