Characteristic classes in stable motivic homotopy theory: TA 1

Examples of topological fiber bundles.

Show that $\mathbb{C}^{n+1} \setminus \{0\} \to \mathbb{CP}^n$ and $U(1) \to SU(1)$ are fiber bundles.

Projective Bundle theorem.

Let $(\mathbb{E}, c)$ be a oriented cohomology theory, and let $p\colon V \to S$ be a vector bundle of rank r. Let $\mathscr{O}_V$ be the canonical line bundle on $\mathbb{P}(V)$. Show that the map

$$\begin{array}{rcl} \bigoplus_{i=0}^{r-1} \mathbb{E}^{**}(S) & \to & \mathbb{E}^{**}(\mathbb{P}(V)) \\ (\lambda_i)_i & \mapsto & \sum_{i=0}^{r-1} p^*(\lambda_i) c_1(\mathscr{O}_V)^i \end{array}$$

is an equivalence. You may first do the the case $V = \mathcal{O}_S^n$ first, using induction. In this case, use that the diagram

$$\begin{array}{ccc} \mathbb{P}^n_S & \xrightarrow{\Delta} & (\mathbb{P}^n_S)^{\wedge n} \\ \downarrow & & \uparrow \\ \mathbb{P}^n_S/\mathbb{P}^{n-1}_S & \xrightarrow{\sim} & (\mathbb{P}^1_S)^{\wedge n} \end{array}$$

commutes.

Splitting principle.

Let $(\mathbb{E}, c)$ be a oriented cohomology theory and let $p : V \to S$ be a vector bundle. Show that there exists a smooth surjective map $\alpha\colon Y \to S$ such that α^* is injective on $\mathbb{E}^{**}(S)$, and $\alpha^{-1}(V) = V \times_S Y \to Y$ is a direct sum of line bundles.

Whitney sum formula.

Let $(\mathbb{E}, c)$ be a oriented cohomology theory. Recall that if $i\colon Z \to X$ is a closed subset of a smooth S-scheme, we will denote by $\mathbb{E}^{**}_Z(X) = \mathbb{E}^{**}(X/(X - Z))$ the cohomology on X with supports in Z. Let Z and T be closed subset of a smooth S-scheme X. Show that the cup product

$$\mathbb{E}^{**}(X) \otimes_{\mathbb{E}^{**}(S)} \mathbb{E}^{**}(X) \to \mathbb{E}^{**}(X)$$

refines to the formula

$$\mathbb{E}^{**}_Z(X) \otimes_{\mathbb{E}^{**}(S)} \mathbb{E}^{**}_T(X) \to \mathbb{E}^{**}_{Z \cap T}(X).$$

For $V \to S$ a vector bundle, denote by $c(V) = \sum_{i=0}^{\infty} c_i(V) t^i \in \mathbb{E}^{**}(S)[[t]]$ the total Chern class of V. Show that if

$$0 \to V \to F \to W \to 0$$

is a short exact sequence of vector bundles on S, then the formula

$$c(F) = c(V)c(W)$$

holds. You may use the splitting principle and the closed subsets $\mathbb{P}(V)$ and $\mathbb{P}(W)$ of $\mathbb{P}(V \oplus W)$.