IAS Summer Collaborators Program 2024 Report:
The Blume-Capel model

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Motivation and problems of interest

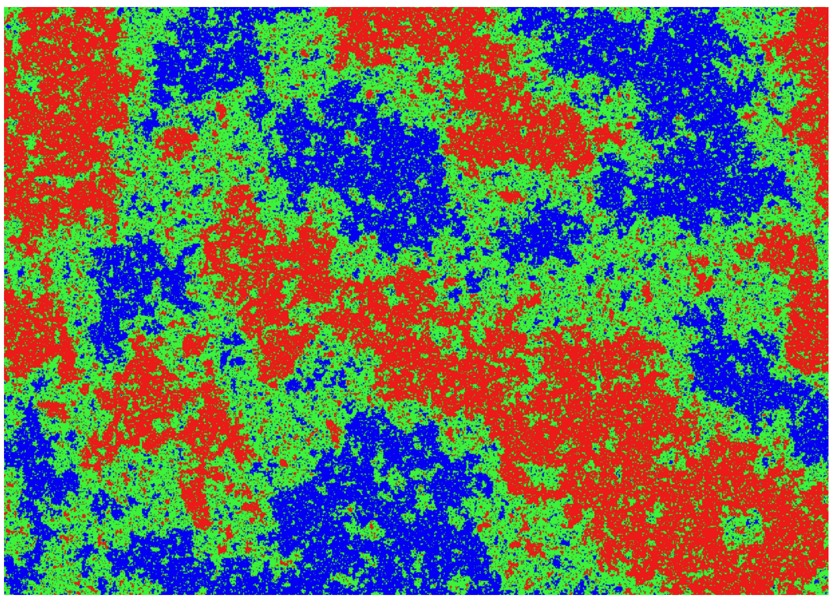

Figure 1: Simulation of the Blume-Capel model at the tricritical point. Blue are +1 spins, red are −1 spins, and green are 0 spins. Image courtesy of Martin Hairer.

The Blume–Capel model was originally introduced by Blume [Blu66] in 1966 to capture an exotic phase transition in UO_2 [FSCO66] and further properties of its phase diagram were established later by Capel [Cap66]. In physics it is an archetypical example of a model exhibiting a *tricritical* point: a boundary point that separates two distinct types of critical behaviour. Mathematically, the model on a finite graph $G = (V, E)$ is given by a probability measure $\mu_{G,\beta,\Delta}$ on spin configurations $\{-1, 0, 1\}^V$ with weight

$$\mu_{G,\beta,\Delta}[\sigma] = \frac{1}{Z_{G,\beta,\Delta}} \prod_{xy\in E} \exp(\beta\sigma_x\sigma_y) \prod_{x\in V} \exp(\Delta\sigma_x^2), \qquad \sigma \in \{-1, 0, 1\}^V.$$

Above, $Z_{G,\beta,\Delta}$ is a normalisation constant, $\beta > 0$ is the inverse temperature which determines the likelihood for spins to align along edges, and $\Delta \in \mathbb{R}$ is the crystal field strength that determines the propensity for a site to be vacant (0) or occupied (±1).

The phase diagram of this model, defined in terms of the *spontaneous magnetization* $m^*(\beta)$, is rich. It has a line of critical points, see Figure 2. One of the most fundamental questions in the study of critical phenomena concerns whether the phase transition is continuous or discontinuous at a critical point. In the Blume–Capel model, along the line of critical points, there is conjecturally a boundary point [GKP24] — the so-called tricritical point $(\Delta_{\text{tric}}, \beta_c(\Delta_{\text{tric}}))$, see Figure 1 — that marks the separation between continuous and discontinuous phase transitions. In a previous work by a subset of the authors, we showed the existence of a (potentially non-unique) tricritcal point. It is, however, predicted that the tricritical point is unique and the phase transition is continuous at it in all dimensions $d \geq 2$. Our work establishes the latter in $d = 2$. The model at its critical points of continuous phase transition is of particular interest. For $\Delta > \Delta_{\text{tric}}$, the critical model is expected to be in the Ising universality class. At the tricritical point, however, the model is predicted to be in a different universality class in all dimensions.

During our time at the IAS, our focus was on critical phenomena in the Blume–Capel model in dimensions $d \geq 4$. For these dimensions, there is already a wide variety of tools and techniques to study critical phenomena in the (nearest-neighbour) Ising model. We were particularly inspired by recent astounding progress using geometric arguments based on percolation representations of the model [ADCS15, ADC21]. These types of arguments allow one to non-perturbatively study models at or near their respective critical points.

Our aims during the program were as follows.

- Building on our previous works [KPP23, GKP24], continue to develop geometric representations for the Blume–Capel model — in particular, construct an analogue to the random current representation of the Ising model.
- Combine the point of view that Blume–Capel is Ising on a random environment together with percolation arguments to analyze the universality class of the model.

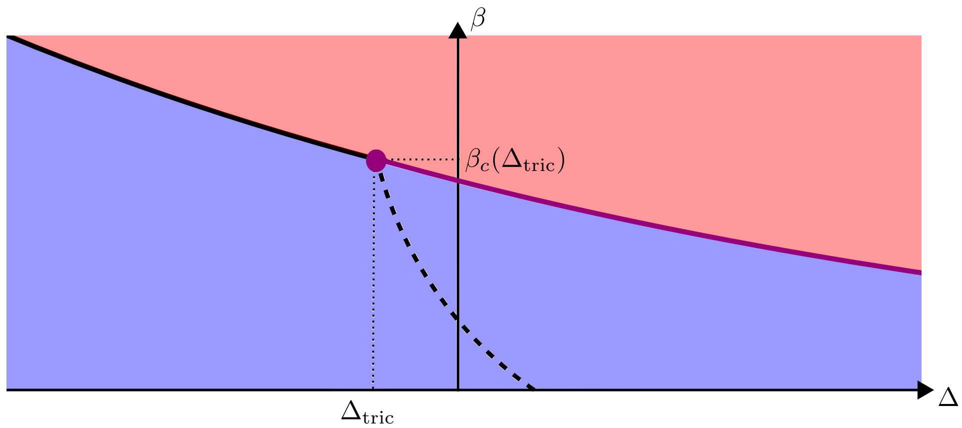

Figure 2: Phase diagram of the Blume-Capel model. Solid line is the critical line of the model. Dashed line is our prediction of the critical line of the underlying site percolation. Red region is supercritical, blue region is subcritical. Purple line are points conjectured to be in Ising universality class. Purple dot is the tricritical point.

Progress and future plans

Understanding the (near) critical behaviour of lattice models is a key challenge in statistical mechanics. A possible approach involves determining the models' *critical exponents*. Performing this task is typically very difficult, as it depends on the unique characteristics of the models and the geometry of the graphs on which they are constructed. A striking observation was made in the case of models defined on the hypercubic lattice $\mathbb{Z}^d$. In sufficiently high dimensions, spin models tend to drastically simplify in the sense that the critical exponents of the model on $\mathbb{Z}^d$ coincide exactly with those expected by considering the model on complete graphs or trees: they adopt their so-called *mean-field* behaviour. The upper critical dimension is the smallest dimension for which this behaviour holds.

One example of such a critical exponent is the magnetization exponent $\boldsymbol{\beta}$, which determines the asymptotics of the spontaneous magnetization as $\beta \downarrow \beta_c$, i.e.

$$m^*(\beta) = (\beta - \beta_c)^{\boldsymbol{\beta}+o(1)}, \tag{1}$$

where $o(1)$ tends to 0 as β tends to β_c.

For the Ising model, the computation of the critical exponents on complete graphs and trees is classical, see [FV17]. For instance, the exponent $\boldsymbol{\beta}$ is known to exist and takes the value $\boldsymbol{\beta} = 1/2$. For the Ising model on $\mathbb{Z}^d$, the upper critical dimension is predicted to be $d_c = 4$ and there has been significant progress towards establishing this. Geometric techniques have played a pivotal role in the analysis of mean-field behaviour in the high dimensional Ising model. On the one hand, Gaussian scaling limits of relevant observables at or near the critical point, which are intimately tied to mean-field behaviour, have been established for $d \geq 4$ by analysis of the model's *random current* percolation representation [Aiz82, ADC21]. On the other hand, the computation of some critical exponents — such as $\boldsymbol{\beta}$ — has been reduced, thanks to a set of differential inequalities (see [Aiz82, AG83, AF86]), to the verification of the so-called *bubble condition* which states that

$$B(\beta_c) := \sum_{x \in \mathbb{Z}^d} \langle \sigma_0 \sigma_x \rangle_{\beta_c}^2 < \infty. \tag{2}$$

Such a condition can be checked in dimensions $d \geq 5$ using the celebrated *infrared bound*, see [FSS76, FILS78]. Furthermore, there has been recent progress towards establishing the remaining mean-field exponents in dimensions $d \geq 5$ [DCP24].

The situation in the case of the Blume–Capel model is much more intricate. The upper-critical dimension of the model at the tricritical point is predicted to be equal to 3: that is to say, the tricritical model exhibits mean-field behaviour in all dimensions $d \geq 3$. On the other hand, away from the tricritical point (but still in the continuous critical regime), the upper critical dimension of the model is predicted to be equal to 4.

The rigorous study of the mean-field Blume–Capel model was initiated in [EOT05]. There, the authors showed via a variational analysis of the free energy that the model exhibits a unique tricritical point. However, they do not compute its critical exponents. During our stay at the IAS, we developed on this analysis and rigorously computed the mean-field exponents of the Blume–Capel model, including at its tricritical point.

Theorem 1. *The magnetization critical exponent of the tricritical Blume–Capel model on the complete graph is given by*

$$\boldsymbol{\beta} = \frac{1}{4}. \tag{3}$$

In particular, the tricritical mean-field Blume-Capel model is in a different universality class from the mean-field Ising model.

In ongoing work, we are extending this analysis to the case of trees. A central difficulty here lies in the fact that we lose the characterization of the tricritical point obtained by [EOT05], and therefore we must give an alternative characterization.

In future work, we aim to address the question of mean-field behaviour in the Blume–Capel model on $\mathbb{Z}^d$ for high dimensions. In that regard, it is tempting to try to extend the approach initiated by Aizenman [Aiz82], and to derive differential inequalities involving the different quantities of interest. As it turns out, for the Blume–Capel model, the so-called *watermelon condition*

$$W(\beta_c) := \sum_{x \in \mathbb{Z}^d} \langle \sigma_0 \sigma_x \rangle_{\beta_c}^3 < \infty \tag{4}$$

should play a pivotal role. See [BLS20] for an illustration of this fact in the perspective of renormalization group analysis.

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