Summer Collaborators Report

I. Beach, H. Contreras Peruyero, E. Griffin,
M. Kerr, R. Rotman, C. Searle.

August 2024

1 Introduction

While at the Institute for Advanced Study we worked in two areas of Riemannian geometry: quantitative geometry and the existence of compact homogeneous Einstein metrics. Our stay resulted in the completion of one paper, the completion of the first draft of another paper, and significant progress on our proposed problem. Additionally, I. Beach submitted a paper [2] and finished the first draft of an additional single-authored paper [1].

2 Quantitative Geometry

2.1 Motivation

The main objective of this part of our work was to study quantitative aspects of solutions to various variational problems, such as periodic geodesics, geodesic loops, geodesic chords, and minimal surfaces, on closed Riemannian manifolds, and to extend this study to the setting of both complete, non-compact manifolds and compact manifolds with boundary. These minimal objects are the natural objects of study of Riemannian geometry with connections to the other fields, such as physics and robotics, and other areas of mathematics, such as dynamical systems. We were, in particular, motivated by the theorem of J. P. Serre establishing the existence of infinitely many geodesics connecting an arbitrary pair of points on a closed Riemannian manifold [14], the "effective" proof of this theorem by A. Schwarz [13], and some related recent progress. We were also motivated by Seifert's conjecture and recent progress in proving the existence of minimal surfaces and orthogonal geodesic chords with free boundaries. Specifically, we have been working on the two problems described below.

2.2 Description of the Problems and our Progress

1. Quantitative Serre's theorem on manifolds with controlled geometry, (H. Alpert, I. Beach, H. Contreras Peruyero, R. Rotman, C. Searle). As mentioned above, a result of J. P. Serre [14] states that given any pair of points on a closed Riemannian manifold M^n, there exist infinitely many geodesics connecting them. It is thus natural to ask if it is possible to estimate the lengths of these geodesics in terms of other geometric parameters of M^n, such as, for example, the diameter of M. It has been proven that for any pair of points on a closed Riemannian manifold M^n there exist at least l geodesics connecting them of length at most $4nl^2d$, where d is the diameter of the manifold [12]. In dimension 2, a bound that is linear in l has also been established [11, 6]. At the moment, the answer to the existence of a curvature-free bound that is linear in l for $n > 2$ seems out of reach. However, we have computed a bound that is linear in l for the length of geodesic segments for any $M^n \in \mathcal{M}^D_{k,v}$, where $\mathcal{M}^D_{k,v}$ denotes the class of manifolds with sectional curvature bounded below by a possibly negative constant k, with volume $\geq v > 0$ and with diameter $\leq D$.

2. On the length of the free boundary orthogonal geodesic chord, (I. Beach, H. Contreras Peruyero, E. Griffin, M. Kerr, R. Rotman, C. Searle. Let N^k be a closed submanifold of dimension k in either
(1) a closed Riemannian manifold M^n; or
(2) a complete Riemannian manifold M^n; or
(3) let N^{n-1} be the boundary of a Riemannian n-disk.
One then asks if there are some constraints on the geometry or topology of M^n and/or N^k that guarantee the existence of a "short" geodesic chord $\gamma : [0, 1] \to M^n$ with endpoints in N that is orthogonal to N at its endpoints. Note that the existence of such a geodesic chord was established in recent papers of Xin Zhao [15], Roberto Giambò, Fabio Giannoni, Paolo Piccione [7], Dongyeong Ko [10], as well as earlier papers by Werner Bos [4] and Joel Hass and Peter Scott [9].

During our stay, I. Beach wrote a paper establishing an effective version of the results of Ko and Hass–Scott of the existence of two simple orthogonal geodesic chords in a 2-disk [1]. As a group we have established a natural geometric condition, which, when satisfied either by M^n or by N^k, allows one to estimate the length of a shortest orthogonal geodesic chord in all of the three cases above. The first draft of the paper is near completion. This project was done at the Institute for Advanced Studies, start to finish.

2.3 Future Plans

There are two natural problems we are planning to consider next.

Problem 1. *Let M be a closed Riemannian manifold with a boundary. The existence of an orthogonal geodesic chord has been proven for Riemannian disks. Can these results be extended to the case where M is of a different topological type? Does there exist at least one geodesic chord orthogonal to the boundary and can one estimate its length?*

Problem 2. *There are numerous recent results proving the existence of minimal submanifolds with free boundary. Can one similarly estimate their size in terms of the parameters of the ambient space?*

3 Existence of Compact Homogeneous Einstein Metrics

3.1 Motivation

A Riemannian manifold (M, g) is said to be *Einstein* if it has constant Ricci curvature, that is, $Ric_g(X, Y) = \lambda g(X, Y)$. Our work is in the setting of compact homogeneous spaces, spaces for which a compact Lie group G acts transitively by isometries, so that $M \cong G/H$.

Einstein metrics have a variational characterization: the (G-invariant) Einstein metrics are exactly the critical points of the scalar curvature functional $s(g)$ on unit volume (homogeneous) metrics. By the work of Böhm [3] and later, Graev [8], we have topological tools based purely on algebraic data for $H < G$. Namely, there is a simplicial complex $\Delta_{G/H}$, or the nerve $X_{G/H}$, that detects when G-invariant Einstein metrics *must* exist for global reasons, identifying non-contractibility of the subset of the domain of the scalar curvature functional where $s(g)$ is sufficiently large. For a compact homogeneous space G/H, if $\Delta_{G/H}$ or $X_{G/H}$ is non-contractible, then G/H admits a G-invariant Einstein metric. However, there are many compact homogeneous spaces where $\Delta_{G/H}$ and $X_{G/H}$ are contractible, and yet admit one or more G-invariant Einstein metrics.

3.2 Description of the Problem and Our Progress

We denote by $\mathcal{N}^c_{0,k}$ the class of compact homogeneous manifolds G/H such that $\mathrm{rk}\,(G) - \mathrm{rk}\,(H) = 0$, the simplicial complex $\Delta_{G/H}$ is contractible, and G/H has k irreducible summands in the isotropy representation. Our project is to consider the set of all compact homogeneous spaces G/H for which the isotropy representation has three or four irreducible summands, and the nerve or simplicial complex *does not* guarantee any Einstein metrics, and identify which of those spaces admit at least one G-invariant Einstein metric. Ultimately, we want to identify an underlying algebraic marker that corresponds precisely to existence of invariant Einstein metrics in this case.

For each compact simple Lie group G, all the full rank subgroups H are known. Furthermore, for each G/H with $\mathrm{rk}\,(G) = \mathrm{rk}\,(H)$, the nerve and the simplicial complex are homeomorphic, and their contractibility or non-contractibility has been computed by Rausse [5]. When G is a *classical* Lie group, there are no homogeneous spaces G/H with $\mathrm{rk}\,(G) = \mathrm{rk}\,(H)$, contractible $\Delta_{G/H}$, and three irreducible summands in the isotropy representation. However, when G is an *exceptional* Lie group, there is precisely one such example: $\mathcal{N}^c_{0,3}$ contains only $\mathrm{F}_4 \,/\, \mathrm{Spin}(5)\,\mathrm{Spin}(4)$, a Grassmann bundle over the Cayley projective plane, admitting a unique F_4-invariant Einstein metric up to homothety. This brings us to the following question.

Problem 3. *Which of the manifolds in $\mathcal{N}^c_{0,4}$ admit a G-invariant homogeneous Einstein metric?*

During our time at IAS, we have been able to identify the full list of homogeneous spaces in $\mathcal{N}_{0,k}$ with $k \leq 4$, and we have several preliminary results. In particular, we have calculated the critical points of the scalar curvature functional for all of the examples in $\mathcal{N}^c_{0,4}$ for which G is a classical Lie group or $G \in \{\mathrm{G}_2, \mathrm{F}_4, \mathrm{E}_6\}$. We have been able to show that each such G/H admits an Einstein metric. Finally, we have also made significant progress for the remaining cases, where $G \in \{E_7, E_8\}$, and hope to finish these quickly.

3.3 Future Plans

We additionally suspect that an answer to Problem 3 may give us sufficient information to identify further algebraic characteristics that differentiate between homogeneous spaces with contractible nerves that do admit Einstein metrics and those that do not. That is, we hope to be able to answer to the following:

Problem 4. *Identify an algebraic characteristic of $H < G$ that determines the existence of a G-invariant Einstein metric on G/H for for $M \in \mathcal{N}^c_{0,4}$, and possibly also when there are more than 4 summands.*

4 Acknowledgements

The support of the Institute for Advanced Study was a significant boost to our research activities. The ability to work in-person in the stimulating environment facilitated by the IAS Summer Collaborators Program has enabled us both to conclude some existing themes and to initiate new avenues of enquiry. We look forward to acknowledging this support in the resulting publications.

References

[1] I. Beach. Short simple orthogonal geodesic chords on a 2-disk with convex boundary, 2024. In preparation.

[2] I. Beach. Short simple geodesic loops on a 2-sphere, 2024. URL `https://arxiv.org/abs/2407.12673`.

[3] C. Böhm. Homogeneous Einstein metrics and simplicial complexes. *J. Differential Geom.*, 67(1):79–165, 2004. ISSN 0022-040X,1945-743X. URL `http://projecteuclid.org/euclid.jdg/1099587730`.

[4] W. Bos. Kritische sehenen auf Riemannischen elementarraumstücken. *Mathematische Annalen*, 151(5):431–451, 1963.

[5] C. C. Rauße. *Simplicial complexes of compact homogeneous spaces.* PhD thesis, Münster, 2016.

[6] H. Y. Cheng. Curvature-free linear length bounds on geodesics in closed Riemannian surfaces. *Transactions of the American Mathematical Society*, 375(7):5217–5237, 2022.

[7] R. Giambó, F. Giannoni, and P. Piccione. Multiple orthogonal geodesic chords and a proof of seifert's conjecture on brake orbits, 2022. URL `https://arxiv.org/abs/2002.09687`.

[8] M. M. Graev. The existence of invariant Einstein metrics on a compact homogeneous space. *Trans. Moscow Math. Soc.*, pages 1–28, 2012. ISSN 0077-1554,1547-738X. doi: 10.1090/s0077-1554-2013-00202-2. URL `https://doi.org/10.1090/s0077-1554-2013-00202-2`.

[9] J. Hass and P. Scott. Shortening curves on surfaces. *Topology*, 33(1):25–43, 1994.

[10] D. Ko. Existence and Morse index of two free boundary embedded geodesics on Riemannian 2-disks with convex boundary, 2023. URL `https://arxiv.org/abs/2309.09896`.

[11] A. Nabutovsky and R. Rotman. Linear bounds for lengths of geodesic segments on Riemannian 2-spheres. *Journal of Topology and Analysis*, 05 (04):409–438, 2013.

[12] A. Nabutovsky and R. Rotman. Length of geodesics and quantitative Morse theory on loop spaces. *Geometric and Functional Analysis*, 23:367–414, 2013.

[13] A. Schwarz. Geodesic arcs on Riemann manifolds. *Uspekhi Matematicheskikh Nauk*, 13(6):181–184, 1958.

[14] J.-P. Serre. Homologie singulière des espaces fibrés. *Annals of Mathematics*, 54(3):425–505, 1951.

[15] X. Zhou. On the free boundary min–max geodesics. *International Mathematics Research Notices*, 2016(5):1447–1466, 06 2015.