Short introduction to higher Berry phase

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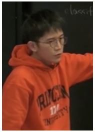
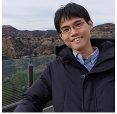
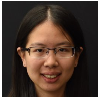
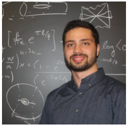
Outline/Acknowledgments

- Introduction (regular Berry phase and others)
- Higher Berry phase using MPS
- Higher dimensions

Shuhei Ohyama (RIKEN)

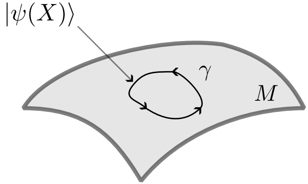


Also: Bowei Liu (Princeton), Yuya Kusuki (Kyushu U), Yuhan Liu (Max Planck), Ramanjit Sohal (Princeton), Jonah Kudler-Flam (IAS, Princeton)

Regular Berry phase

- As the higher Berry phase is a generalization of regular Berry phase, let's start by reviewing the regular Berry phase
- In a typical setup, we consider a quantum state $|\psi\rangle$ that depends smoothly on some parameter $X = (X^1, X^2, \cdots)$. X can change over some manifold M ("parameter space").



- Berry phase is the phase accumulated by $|\psi\rangle$ as we change X smoothly or adiabatically.

– The Berry phase along a loop γ can be calculated as

$$i \oint_\gamma dt \, \langle\psi(X(t))| \tfrac{d}{dt} |\psi(X(t))\rangle = i \oint_\gamma dt \, \underbrace{\langle\psi(X))| \frac{\partial}{\partial X^\mu} |\psi(X)\rangle}_{\equiv \mathcal{A}_\mu(X)} \tfrac{dX^\mu}{dt}$$

$$= i \oint_\gamma \mathcal{A}_\mu dX^\mu = i \oint_\gamma \mathcal{A}$$

– The Berry phase is invariant under (small) gauge transformation, $|\psi(X)\rangle \to e^{i\phi(X)}|\psi(X)\rangle$, $\mathcal{A}_\mu \to \mathcal{A}_\mu + \partial_\mu \phi$:

$$\oint_\gamma \mathcal{A} \to \oint_\gamma (\mathcal{A} + d\phi) = \oint_\gamma \mathcal{A} + \oint_{\partial\gamma} \phi = \oint_\gamma \mathcal{A}$$

– $\mathcal{A}_\mu$ couples to a conserved current (particle trajectory) in parameter space,

$$i \oint_\gamma \mathcal{A} = i \int d^D X \, \mathcal{A}_\mu j^\mu$$

$$j^\mu(X) = \int dt \, \delta^D(X - X(t)) \dot{X}^\mu$$

Topological classification

– We can also consider a topological classification of parameterized families of $|\psi(X)\rangle$ over M.

– Each family can be thought of as defining a complex line bundle. Here, a fibre corresponds to a quantum state (projector) or its phase ambiguity part.

– Complex line bundles are classified by Chern numbers and Chern classes – these are topological invariants of vector bundles (line bundles).

– There are various ways to think about them. E.g.,
  - In terms of curvature
  - In terms of transition functions

Curvature

- In the first viewpoint, we consider Berry curvature,

$$\mathcal{F} = d\mathcal{A} = \partial_\mu \mathcal{A}_\nu dX^\mu dX^\nu = \frac{1}{2}\mathcal{F}_{\mu\nu} dX^\mu dX^\nu$$

- The Berry curvature $\mathcal{F}/2\pi i$ can be integrated over any 2-cycle in M (with integer coefficients). By Dirac's argument, the integral is quantized to be an integer independent of the connection.

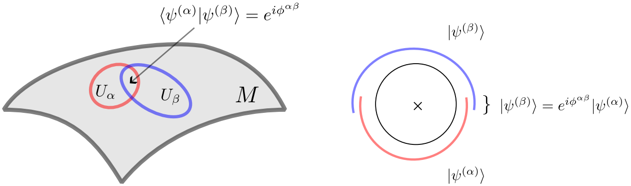
- On the other hand, $\mathcal{F}/2\pi i$ is a closed, and globally defined 2-form (does not depend on the gauge choice for patches).

- Therefore, it defines an element in the 2nd cohomology group, $H^2(M, \mathbb{R}) =$ space of closed 2-forms/space of exact 2-forms.

- Because of the Dirac quantization, it is natural to consider Chern classes as an element of cohomology group with integer coefficients, $[\mathcal{F}/2\pi i] \in H^2(M, \mathbb{Z})$.

Transition functions

- Wu-Yang's approach to magnetic monopoles
- Consider to patch M, so that for each patch, we can define a smooth gauge of wavefunction.



- Within the intersection $U_\alpha \cap U_\beta$, two wave functions, one from each patch, must be physically equivalent, and related by a gauge transformation

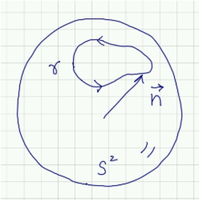
$$|\psi_\alpha\rangle = e^{i\phi_{\alpha\beta}}|\psi_\beta\rangle, \qquad e^{i\phi_{\alpha\beta}}: \text{ transition function}$$

- The data $(\{U_\alpha\}, \{e^{i\phi_{\alpha\beta}}\})$ topologically defines a complex line bundle.
- $[e^{i\phi_{\alpha\beta}}] \in \check{H}^1(M, \underline{U(1)}) \simeq H^2(M, \mathbb{Z})$.

Examples

- Single spin with $M = S^2$

$$|\psi(\vec{n})\rangle = e^{i\chi} \begin{pmatrix} e^{-i\phi/2}\cos\theta/2 \\ e^{+i\phi/2}\sin\theta/2 \end{pmatrix}$$



- Thouless pump [Thouless (83)]

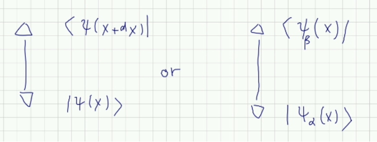
$$H = \sum_i \left[-(J+\delta) f_i^\dagger d_i - (J-\delta) f_i^\dagger d_{i+1} + h.c. + \Delta(f_i^\dagger f_i - d_i^\dagger d_i) \right]$$

$$Ch = \int_0^T dt \int_{-\pi}^{\pi} dk\, \mathcal{F}(k,t)/2\pi i = \text{integer}$$

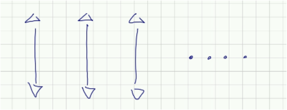
[C.f. Experiment; Nakajima et al (16)]

Introduction to higher Berry phase

- So far: Berry phase for a single quantum spin; Berry phase is associated with a wave function overlap,

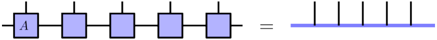


- In many-body physics, we have many spins. So we may consider:



  But such a product state is too boring.

- We will consider more correlated states, described by MPS:

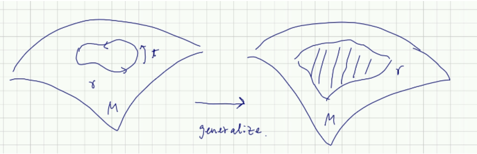


- Faithful representation of short-range entangled or invertible states

- Roughly, we generalize $X(t) \to X(t,\sigma)$ and $|\psi(X(t))\rangle \to |\psi[X(t,\sigma)]\rangle$.
- Now we can define "string current", which generalizes "particle current":

$$j^\mu(X) = \int dt \delta^D(X - X(t)) \frac{dX^\mu}{dt}$$
$$\to j^{\mu\nu}(X) = \int dt d\sigma \delta^D(X - X(t)) \frac{1}{2} \left[\frac{\partial X^\mu}{\partial t} \frac{\partial X^\nu}{\partial \sigma} - \frac{\partial X^\nu}{\partial t} \frac{\partial X^\mu}{\partial \sigma} \right]$$

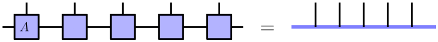
- Such string current can couple to a two-form gauge field $B_{\mu\nu}$:
- As before, we can consider $i \int_M B_{\mu\nu} j^{\mu\nu} = i \int_\gamma B_{\mu\nu} dX^\mu dX^\nu$.
  There is a 1-form gauge invariance, $B_{\mu\nu} \to B_{\mu\nu} + \partial_\mu \xi_\nu - \partial_\nu \xi_\mu$.



- So this is what we would expect; We will have 2-form connection.

Gapped states in (1+1)d and higher Berry phase

- Short-range entangled or invertible states in (1+1)d
- Can be expressed using matrix product states (MPS)



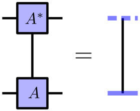
[Previous and related works on higher Berry phase:

Kitaev (2019); Cordova-Freed-Lam-Seiberg (19); Kapustin-Spodyneiko (20); Kapustin-Sopenko (22); Hsin-Kapustin-Thorngren (20); Artymowicz-Kapustin-Sopenko (23); Choi-Ohmori (22); Shiozaki (21); Wen-Qi-Beaudry-Moreno-Pflaum-Spiegel-Vishwanath-Hermele (21); Beaudry-Hermele-Moreno-Pflaum-Qi-Spiegel (23); Ohyama-Shiozaki-Sato (22); Ohyama-Terashima-Shiozaki (23); Beaudry-Hermele-Moreno-Pflaum-Qi-Spiegel (23); Qi-Stephen-Wen-Spiegel-Pflaum-Beaudry-Hermele (23); Shiozaki-Heinsdorf-Ohyama (23); Sommer-Wen-Vishwanath (24); Sommer-Vishwanath-Wen (24) ...]

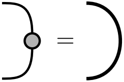
MPS for short-range entangled states

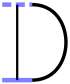
- We consider normal MPS, for which the transfer matrix has a unique largest eigenvalue.

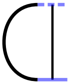
- Transfer matrix:

$T_A =$ 

and its left and right actions:

$M =$ 

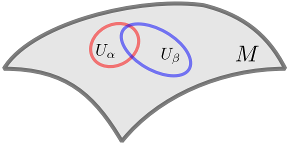
$T_A \cdot M =$ 

$M \cdot T_A =$ 

- We denote the unique dominant left and right eigenvectors as Λ^L and Λ^R. By taking the right canonical form, we can take $\Lambda^R = 1$ as the most dominant right eigenvector with eigenvalue 1. Biz, $\sum_s A^s \cdot 1 \cdot A^{s\dagger} = 1$

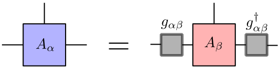
Double intersection

- Now, as before, we patch M, $\{U_\alpha\}$.



- For each patch, MPS representation $A_\alpha(X)$ is smooth over U_α.
- When two patches intersect, two MPS represent the same physical state.
- The fundamental theorem then states that the MPS are related as

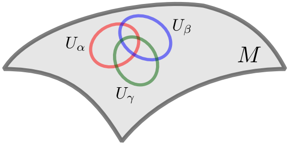
$$A^s_\alpha = g_{\alpha\beta} A^s_\beta g^\dagger_{\alpha\beta} e^{i\chi}$$



This relation should play a simliar role as $|\psi_\alpha\rangle = e^{i\phi_{\alpha\beta}}|\psi_\beta\rangle$.
We call $g_{\alpha\beta}$ transition function.

Triple intersection

- When three patches intersect, on $U_\alpha \cap U_\beta \cap U_\gamma$, we can consider three transition functions, $g_{\alpha\beta}$, $g_{\beta\gamma}$, $g_{\gamma\delta}$ (with $g_{\beta\alpha} = g^\dagger_{\alpha\beta}$ etc.)



- Recall that previously, we had $e^{i\phi_{\alpha\beta}}$, $e^{i\phi_{\beta\gamma}}$, $e^{i\phi_{\gamma\alpha}}$.
  They satisfy the cocycle condition $e^{i\phi_{\alpha\gamma}} = e^{i\phi_{\alpha\beta}} e^{i\phi_{\beta\gamma}}$ since we can "relate" $|\psi_\gamma\rangle$ and $|\psi_\alpha\rangle$ in two different ways:

$$|\psi_\alpha\rangle = e^{i\phi_{\alpha\gamma}} |\psi_\gamma\rangle \quad : \text{direct way}$$

$$|\psi_\alpha\rangle = e^{i\phi_{\alpha\beta}} |\psi_\beta\rangle = e^{i\phi_{\alpha\beta}} e^{i\phi_{\beta\gamma}} |\psi_\gamma\rangle \quad : \text{indirect way}$$

  These should be consistent, and hence we have the cocycle condition.

Triple intersection

– Let us repeat this argument, and relate A^s_γ and A^s_α in two different ways:

$$A^s_\alpha = g_{\alpha\gamma} A^s_\gamma g^\dagger_{\alpha\gamma} \quad : \text{direct way}$$
$$A^s_\alpha = g_{\alpha\beta} A^s_\beta g^\dagger_{\alpha\beta} = g_{\alpha\beta} g_{\beta\gamma} A^s_\gamma g^\dagger_{\beta\gamma} g^\dagger_{\alpha\beta} \quad : \text{indirect way}$$

– Once again, we demand the consistency; however, there is a $U(1)$ ambiguity,

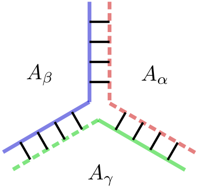
$$g_{\alpha\beta} g_{\beta\gamma} = g_{\alpha\gamma} \times c_{\alpha\beta\gamma}$$

"Ambiguity of ambiguity" (g is an ambiguity of A, c is an ambiguity of g)

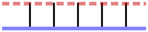
– Recall that $\{e^{i\phi_{\alpha\beta}}\}$ defines an element in $H^1(M, U(1)) \simeq H^2(M, \mathbb{Z})$. Biz, Chern class.

– (We can show that $c_{\alpha\beta\gamma}$ satisfies the cocycle condition. Hence –) Similarly, $\{c_{\alpha\beta\gamma}\}$ defines an element in $H^2(M, U(1)) \simeq H^3(M, \mathbb{Z})$. This class is called the Dixmir-Douady class, which is a topological invariant.

- The transition function $e^{i\phi_{\alpha\beta}}$ can be "extracted" from the wavefunction overlap $\langle\psi_\alpha|\psi_\beta\rangle$ – This is the work of Wu-Yang relating physics (QM with monopole) and mathematics (complex line bundle).
  Is there any physical quantity related to $c_{\alpha\beta\gamma}$?
- As a candidate, let us consider the triple inner product,

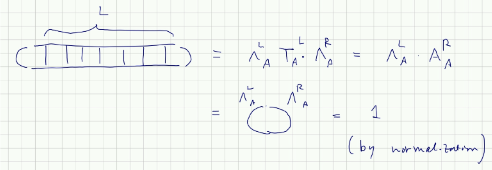


- Recall that the regular inner product:

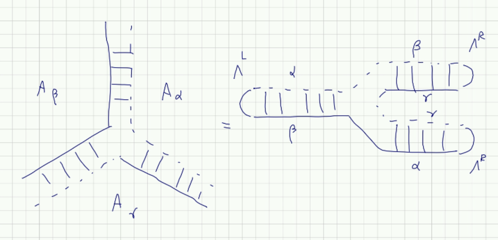


  takes two quantum states (MPS) and spits out one complex number.
  The triple inner product takes three states and spits one number.

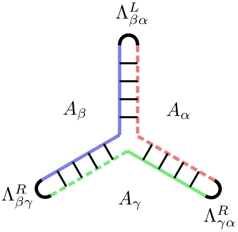
- Now, how can we calculate the triple inner product?
- Recall that the regular inner product can be computed by using the transfer matrix. In the thermodynamic limit, only fixed points of T_A matter. So:



– Similarly, we consider:



– This diagram is evaluated as



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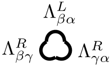


$$= \mathrm{tr}\left(\Lambda^L_{\beta\alpha}\Lambda^R_{\beta\gamma}\Lambda^R_{\gamma\alpha}\right) = \mathrm{tr}\left(\Lambda^L_{\alpha}\hat{g}_{\alpha\beta}1_n\hat{g}_{\beta\gamma}1_n\hat{g}_{\gamma\alpha}\right)$$

$$= \mathrm{tr}\,(\Lambda^L_\alpha)c_{\alpha\beta\gamma}$$

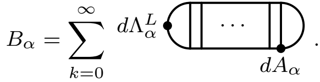
Curvature and connection

- [Kapustin-Spodyneiko (20)]
- [Kapustin-Sopenko (22)] [Artymowicz-Kapustin-Sopenko (23)]
- [Shiozaki-Heinsdorf-Ohyama (23)]
- [Sommer-Wen-Vishwanath (24)] [Ohyama-SR (24)]
- . . .

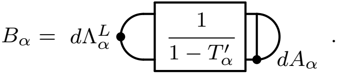
Higher Berry connection

[Sommer-Wen-Vishwanath (24)] [Ohyama-SR (24)]

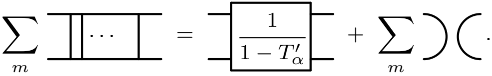
– 2-form connection:



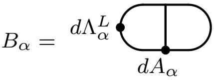
– Can also be written as

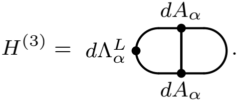


by summing over "ladder diagrams":



– For fixed point MPS:





Example; $M = S^3$

[Wen-Qi-Beaudry-Moreno-Pflaum-Spiegel-Vishwanath-Hermele (21)]

$$H(\vec{w}) = H^{\text{on-site}}(\mathbf{w}) + H^{\text{odd}}(w_4) + H^{\text{even}}(w_4)$$

$$H^{\text{on-site}}(\mathbf{w}) = \sum_{p} (-1)^p \mathbf{w} \cdot \boldsymbol{\sigma}_p ,$$

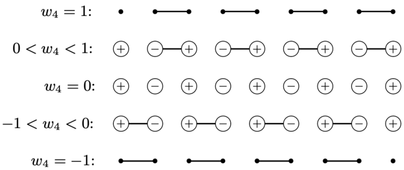
$$H^{\text{odd}}(w_4) = \sum_{p:\text{odd}} g^{\mathrm{N}}(\vec{w}) \boldsymbol{\sigma}_p \cdot \boldsymbol{\sigma}_{p+1} ,$$

$$H^{\text{even}}(w_4) = \sum_{p:\text{even}} g^{\mathrm{S}}(\vec{w}) \boldsymbol{\sigma}_p \cdot \boldsymbol{\sigma}_{p+1} .$$

$$M = S^3 = \{\vec{w} = (\mathbf{w}, w_4) \mid \sum_{\mu=1}^{4} w_\mu^2 = 1\}$$

$$g^{\mathrm{N}}(\vec{w}) = \begin{cases} w_4 & (0 \leq w_4 \leq 1) , \\ 0 & (-1 \leq w_4 \leq 0) , \end{cases}$$

$$g^{\mathrm{S}}(\vec{w}) = \begin{cases} 0 & (0 \leq w_4 \leq 1) , \\ -w_4 & (-1 \leq w_4 \leq 0) . \end{cases}$$



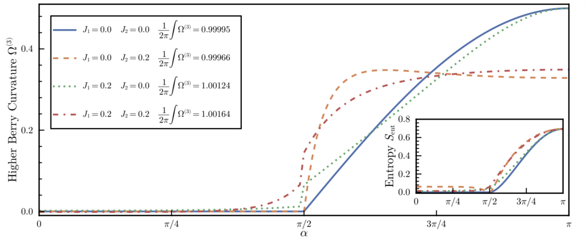
– By an explicit calculation,

$$B^{(2)} = -\frac{i}{2} \cos(t) \cos(\theta) dt \wedge d\phi,$$

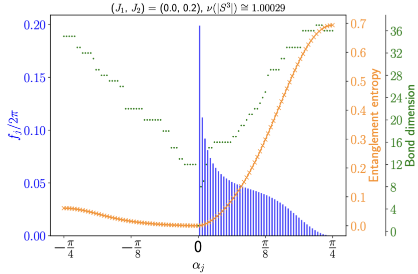
$$\int_{S^3} dB^{(2)} / 2\pi i = -1$$

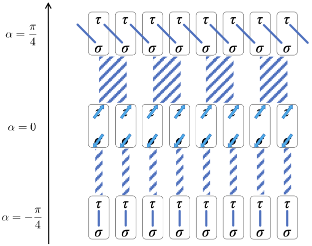
$$(w_4 = \cos(t),\ 0 \leq t \leq \pi),$$

[Sommer-Wen-Vishwanath (24)]



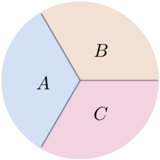
[Shiozaki-Heinsdorf-Ohyama (23)]





Higher dimensions

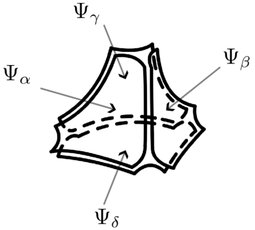
- Let us briefly discuss generalization to (2+1)d. [Ohyama-SR (24); Qi-Stephen-Wen-Spiegel-Pflaum-Beaudry-Hermele1 (23)]
- A natural generalization of triple inner product is quadruple inner product that assigns a complex number for four quantum states. It can be defined as follows.
- We start with (2+1)d invertible states. They can be represented by 2d tensor network, e.g., PEPS of some sort.
- We first divide the 2d space into three regions, A, B, C:

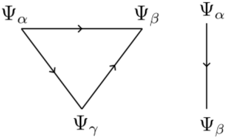


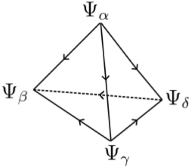
- Consider four states, Ψ_α, Ψ_β, Ψ_γ, Ψ_δ. Each of them are tripartitioned.
- We can now "connect" or "contract" these states with each other.

Qudruple inner product

- It's a bit complicated to draw, but the contraction looks like this:
- A simplified notation: the triple inner product for three bipartite states, $|\Psi\rangle = \sum \psi_{ij} |i\rangle_A |j\rangle_B$.
- In this notation, the regular inner product is simply a line segment:
- The quadruple inner product is defined for four tripartite states, $|\Psi\rangle = \sum_{ijk} \psi_{ijk} |i\rangle|j\rangle|k\rangle$. So, each wave function may be viewed as a trivalent vertex.

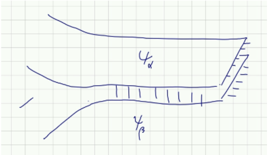


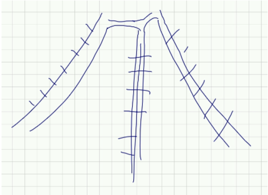




Qudruple inner product

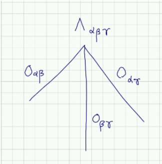
- Let us move on to the "evaluation" of the quadruple product.
- As in the (1+1)d case, it is important to think what happens at infinities.
- For example, two states Ψ_α and Ψ_β "meet": Following the (1+1)d case, we need to contract the boundary indices. So we use Matrix Product Operators (MPO) or Matrix Product Unitaries (MPU).
- This is not the end of the story. We also have corners, where three states meet



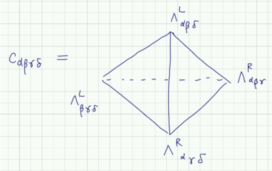


Qudruple inner product

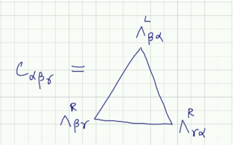
- At the corners, three MPU must meet. So, we also need a tensor to connect different MPUs:



- In the end, the quadruple inner product reduces to a network (tetrahedron) formed by 3-leg tensors (and MPU):

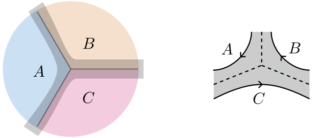


- This is analogous to (1+1)d case where the triple inner product reduces to a triangle of fixed point tensors.



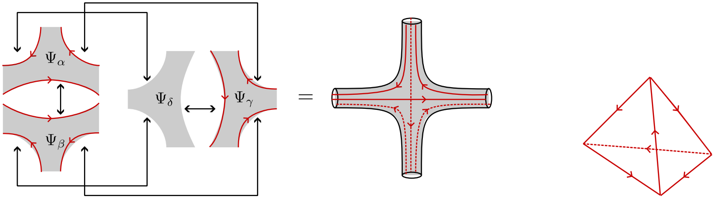
Comments

- Concrete implementation and calculations using semi injective PEPS [Molnar-Ge-Schuch-Cirac (18)]
- higher Berry curvature [Sommer-Vishwanath-Wen (24)]
- Bulk-boundary correspondence: [Bowei Liu-Kusuki-Ohyama-SR, to appear]



[C.f. Yuhan Liu-Kusuki-Sohal-Kudler-Flam-SR (23)]

Quadruple inner product can be represented by a CFT partition function in (1+1)d:



Summary/Comments

- Constant-rank vs non-constant rank
- Underlying mathematical structure: gerbe, a generalization of a principal bundle
  [Ohyama-SR (23)] [Qi-Stephen-Wen-Spiegel-Pflaum-Beaudry-Hermele1 (23)]
- Application to symmetry-protected topological phases
- Flow of Berry curvature, bulk-boundary correspondence.
  [C.f. Kapustin-Spodyneiko (20), Wen-Qi-Beaudry-Moreno-Pflaum-Spiegel-Vishwanath-Hermele (21)]