PCMI LECTURE NOTES: MOTIVIC EXPLORATIONS IN ENUMERATIVE GEOMETRY

SABRINA PAULI

Abstract. These are lecture notes for the PCMI 2024 Graduate Summer School for the mini-workshop on motivic explorations in enumerative geometry.

Motivic homotopy theory allows to do enumerative geometry over an arbitrary field, which leads to additional arithmetic and geometric information. The goal of the mini-workshop is to explain why and how this works. We will also provide a toolbox for solving enumerative geometry problems in this setting, including the use of tropical geometry.

We start with two classic examples in enumerative geometry, namely Bézout's theorem and the count of lines on a cubic surface. We then explain how to solve these problems, first over the complex and real numbers, and then over an arbitrary field, using the $\mathbb{A}^1$ degree from motivic homotopy theory. Then we introduce tropical geometry, more precisely we focus on tropical plane curves and show how they can be used to prove Bézout's theorem for curves over an arbitrary field. Finally, we discuss tropical correspondence theorems arising from joint work with Jaramillo Puentes and joint work in progress with Jaramillo Puentes-Markwig-Röhrle.

Contents

Expected background and references:

- Some basic notions in intersection theory, characteristic classes and the implications to enumerative geometry [EH16, Chapters 1-6], [MS05].
- The definition of the unstable motivic homotopy category. A well written survey article is [WW20].

1. Enumerative geometry over different fields

Content of the Lecture:

- First examples in enumerative geometry: Bézout's theorem and the count of lines on a smooth cubic surface.
  Literature: [EH16]
- The real story: The Poincaré-Hopf theorem and its applications to Bézout's theorem and the real signed count of lines on a smooth cubic surface.
  Literature: [MS05, Mil65, OT14, FK13]
- The Poincaré-Hopf theorem in motivic homotopy theory.
  Literature: [KW21, BW23, McK21]

Date: April 2024.

In enumerative geometry, we want to count geometric objects, such as points, lines, curves,... that satisfy certain algebraic conditions. An example of a result in enumerative geometry is one that everyone learns in the first course in algebraic geometry:

Example 1.1 (Bézout's theorem)**.** For $i = 1, \dots, n$ let $H_i = V(F_i) \subset \mathbb{P}^n_{\mathbb{C}}$ be hypersurfaces defined by a homogeneous polynomial F_i of degree d_i, such that $H_1 \cap \dots \cap H_n$ is zero-dimensional. Then

$$\sum_{x \in H_1 \cap \ldots \cap H_n} \mathrm{mult}_x(H_1, \dots, H_n) = d_1 \cdot \ldots \cdot d_n.$$

In words, counted with intersection multiplicity, there are always $d_1 \cdot \ldots \cdot d_n$ intersection points.

Here is another famous example.

Example 1.2 (Lines on a smooth cubic surface)**.** Let $V(f) \subset \mathbb{P}^3_{\mathbb{C}}$ be a smooth cubic surface (so f is a homogeneous polynomial of degree 3). Then

$$\#\{\ell \subset V(f) : \ \ell \text{ a line}\} = 27$$

regardless of the choice of smooth cubic surface.

Often one can solve problems in enumerative geometry by "linearization", i.e., by computing degrees of chern classes. We illustrate this for the two examples above.

Example 1.3 (Bézout's theorem continued)**.** Let $p\colon V := \mathcal{O}(d_1) \oplus \ldots \oplus \mathcal{O}(d_n) \to \mathbb{P}^n_{\mathbb{C}} =: X$. Then $\operatorname{rank} V = \dim X$ and thus a general section of this vector bundle has finitely many zeros. Now notice that an n-tuple $(F_1, \dots, F_n)$ of homogeneous polynomials of degree $d_1, \dots, d_n$ defines a section $\sigma_{F_1,\dots,F_n}$ of $p\colon V \to X$ and the zeros of the section are exactly the intersections of the hypersurfaces $H_1, \dots, H_n$. Thus

$$\sum_{x \in H_1 \cap \ldots \cap H_n} \mathrm{mult}_x(H_1, \dots, H_n) = \sum_{\text{zeros } z \text{ of } \sigma_{F_1,\dots,F_n}} \mathrm{mult}(z)$$

where $\mathrm{mult}(z)$ is the multiplicity of the zero z of $\sigma_{F_1,\dots,F_n}$. For a general section of a vector bundle of rank n over an n-dimensional scheme, the top chern class is represented by the 0-cycle of the zero locus of a general section and thus $\sum_{\text{zeros } z \text{ of } \sigma_{F_1,\dots,F_n}} \mathrm{mult}(z) = \deg c_n(V)$ where $c_n(V)$ is the top chern class. By the Whitney sum formula

$$\deg c_n(V) = \deg c_1(\mathcal{O}(d_1)) \cdot \ldots \cdot \deg c_1(\mathcal{O}(d_n)) = d_1 \cdot \ldots \cdot d_n.$$

Example 1.4 (Lines on a smooth cubic surface continued)**.** Let $\mathrm{Gr}(2,4) = \mathbb{G}(1,3)$ be the Grassmannian of lines in $\mathbb{P}^3_{\mathbb{C}}$, that is $[\ell] \in \mathrm{Gr}(2,4)$ corresponds to a line $\ell \subset \mathbb{P}^3_{\mathbb{C}}$. Further let $\mathcal{S} \to \mathrm{Gr}(2,4)$ the tautological bundle. Let $p\colon V = \mathrm{Sym}^3 \mathcal{S}^* \to \mathrm{Gr}(2,4) =: X$ be the vector bundle given by the third symmetric power of the dual tautological bundle, that is the vector bundle with fibers

$$(\mathrm{Sym}^3 \mathcal{S}^*)_{[\ell]} = \text{homogeneous degree 3 polynomials on } \ell.$$

Then a smooth cubic surface $V(f) \subset \mathbb{P}^3_{\mathbb{C}}$ defines a section σ_f of $p\colon V \to X$, namely the section defined by restricting f to the line, i.e. $\sigma_f([\ell]) = f|_\ell$. Then

$$\ell \subset V(f) \Leftrightarrow f|_l = 0 \Leftrightarrow \sigma_f([\ell]) = 0$$

and so

$$\#\{\ell \subset V(f) : \ \ell \text{ a line}\} = \#\{\text{zeros of } \sigma_f\}$$

One can show that for smooth cubic surfaces all zeros of σ_f are simple, i.e. they are zeros of multiplicity 1, and the number of zeros of a general section equals the degree of the top chern class $\deg c_4(V)$. One can use Schubert calculus to compute $\deg c_4(V) = 27$.

In real enumerative geometry one is interested in the geometric objects which are defined over the real numbers.

Example 1.5 (Real lines on a smooth cubic surface)**.** On a smooth cubic surface defined by a polynomial f with real coefficients, there can be 3, 7, 15 or 27 real lines depending on the choice of cubic surface [Sch63].

This example shows that we lose *invariance* when we count over non-algebraically closed fields. We already observed this in high school: A polynomial $p \in \mathbb{R}[x]$ (in one variable) of degree d has d zeros (if you count with multiplicity) defined over $\mathbb{C}$, but over $\mathbb{R}$ it can happen that there are none if d is even, or only 1 if d is odd, but it can also happen that all zeros are real. However, we still want to have an invariant count over non-algebraically closed fields.

Idea 1.6. Replace the top chern class in the examples with the Euler class $e(V)$.

Example 1.7 (Real lines on a smooth cubic surface)**.** There is a "real version" of the vector bundle from Example 1.4 $p\colon \operatorname{Sym}^3 \mathcal{S}^* \to \operatorname{Gr}_{\mathbb{R}}(2,4)$ where $\operatorname{Gr}_{\mathbb{R}}(2,4)$ is the real Grassmann manifold of 2-planes in $\mathbb{R}^4$. The degree of the Euler class of this real vector bundle equals

$$\deg e(\operatorname{Sym}^3 \mathcal{S}^* \to \operatorname{Gr}(2,4)) = 3.$$

But what does the 3 tell us? The next definition 1.8 and the Poincaré-Hopf theorem 1.10 will tell us. For this recall the degree of a map from the n-sphere to the n-sphere is from algebraic topology. Let $f\colon S^n \to S^n$ be continuous. Then f induces $f_*\colon \widetilde{H}_n(S^n) \cong \mathbb{Z} \to \widetilde{H}_n(S^n) \cong \mathbb{Z}$ a map in the nth reduced singular homology. The degree of f is the image of 1 under this map $\deg f = f_*(1)$. When $f \simeq g\colon S^n \to S^n$ are homotopic, they induce the same map in homology and thus have the same degree. So we get

$$\deg\colon [S^n, S^n] \to \mathbb{Z} \tag{1}$$

where $[S^n, S^n]$ denotes the homotopy classes of maps from S^n to S^n.

Definition 1.8. Let $p\colon V \to X$ be a real vector bundle of rank n over a real smooth n-manifold.

- Let T_X be the tangent bundle of X. A *relative orientation* of $p\colon V \to X$ is an isomorphism of line bundles $\rho\colon \mathcal{H}om(\det T_X, \det V) \cong \underline{\mathbb{R}}$, where $\underline{\mathbb{R}}$ denotes the trivial line bundle. We say that a vector bundle $p\colon V \to X$ is *relatively orientable* if a relative orientation exists, and that the vector bundle is *relatively oriented* if we have chosen a relative orientation.
- Assume $p\colon V \to X$ is relatively oriented with relative orientation ρ. Let $x \in X$ and choose a small neighborhood U of x such that there is a local trivialization of $V|_U$. We say that local coordinates around x and a trivialization $V|_U \cong U \times \mathbb{R}^n$ are *compatible* with ρ if the distinguished element of $\mathcal{H}om(\det T_X|_U, \det V|_U)$ taking the distinguished basis of $\det T_X|_U$ to $\det V|_U$ is sent to 1 by ρ.
- Assume $p\colon V \to X$ is relatively oriented by ρ and $\sigma\colon X \to V$ is a section. Let x be an isolated zero of σ. Locally around x we can choose coordinates and a trivialization of V compatible with ρ. Then the section σ is given locally by $\sigma\colon \mathbb{R}^n \to \mathbb{R}^n$ (x corresponds to 0 in the source) in these coordinates and with this trivialization. The *local index* $\operatorname{ind}_x \sigma$ of σ at x is the local degree of σ at 0, i.e. it is determined by

$$\sigma_*\colon H_n(\mathbb{R}^n, \mathbb{R}^n \setminus \{0\}) \cong \mathbb{Z} \xrightarrow{\cdot \operatorname{ind}_x \sigma} H_n(\mathbb{R}^n, \mathbb{R}^n \setminus \{0\}) \cong \mathbb{Z}$$

or equivalently it is the degree of

$$\overline{\sigma}\colon {}^{B_\epsilon(x)}/_{\partial B_\epsilon(x)} \simeq S^n \to {}^{B_{\epsilon'}(0)}/_{\partial B_{\epsilon'}(0)} \simeq S^n$$

where $B_\epsilon(x) \subset X$ is a small ball around x and $B_{\epsilon'}(0) \subset \mathbb{R}^n$ is a small ball around 0.

Remark 1.9. *In case x is a simple zero of σ, that is a zero with multiplicity 1, then locally σ is a homeomophism and thus $\operatorname{ind}_x \sigma \in \{\pm 1\}$. In this case $\operatorname{ind}_x \sigma = \det \operatorname{Jac} \overline{\sigma}(0)$ with $\overline{\sigma}$ as in Definition 1.8.*

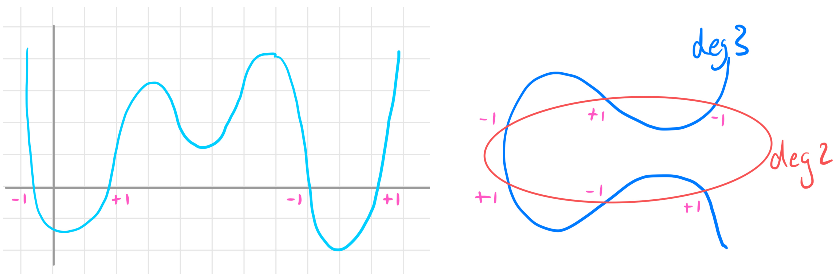


FIGURE 1. Real Bézout for $n = 1$ and $n = 2$

Theorem 1.10 (Poincaré-Hopf theorem)**.** *Let X be a real smooth closed n-manifold and $p\colon V \to X$ be a real relatively oriented vector bundle of rank n. Let σ be a section of $p\colon V \to X$ with only isolated zeros. Then*

$$\deg e(V) = \sum_{\text{zeros } x \text{ of } \sigma} \operatorname{ind}_x \sigma$$

Example 1.11 (Lines on a smooth cubic surface continued)**.** If the cubic surface $V(f)$ is smooth, then all zeros of the corresponding section σ_f are simple. So the local indices are either $+1$ or -1. That is, we get

$$3 = \deg e(V) = \sum_{\ell \subset V(f)} \operatorname{ind}_{[\ell]} \sigma_f$$

an invariant signed count of real lines on a smooth real cubic surface. In fact, one can describe the sign of a real line intrinsically, i.e. without choosing a local trivialization and local coordinates, and this has already been done by Segre [Seg42]. Note that a real line $\mathbb{R}P^1 = S^1$ is a circle. If you follow the normal vector pointing out of the cubic surface along the circle, you either make a full turn or not. In the first case the line has local index -1 and in the second case it has local index $+1$. So the local index depends on how the line is embedded in the cubic surface.

Example 1.12 (Bézout continued)**.** The real analog $p\colon V = \mathcal{O}(d_1) \oplus \ldots \oplus \mathcal{O}(d_n) \to \mathbb{RP}^n$ of the vector bundle from the Bézout example 1.3 is relatively orientable if and only if $\sum d_i \equiv n + 1 \mod 2$.

For example, if $n = 1$, this is the case if d_1 is even. In this case, Bézout's theorem counts the zeros of a polynomial in one variable. The degree of the Euler class of an odd rank bundle is zero. If the derivatives at the zeros do not vanish, then the local index is given by the sign of the derivative by Remark 1.9 and thus the sum of these signs is always zero (see the left picture in Figure 1), but the number of real zeros depends on the polynomial.

For $n = 2$ the vector bundle is relatively orientable if $d_1 + d_2$ is odd. In this case, the local index at a real intersection records the order in which the two curves intersect after orienting them (see the right image in figure 1). It can be computed as the sign of the determinant of the Jacobian at the intersection. Since V has an odd rank summand, applying the Whitney sum formula, we get that the degree of the Euler class of V is zero, and thus the signed count of intersection points is zero.

Question 1.13. *What about other (non-algebraically closed) fields k?*

To answer this question we need to borrow a result from motivic homotopy theory, the analog of the degree (1) we used to define the local index.

Theorem 1.14 (Morel)**.** *Let k be a perfect field. Then there exists a well-defined degree map*

$$\deg^{\mathbb{A}^1} : [\mathbb{P}^n_k/\mathbb{P}^{n-1}_k, \mathbb{P}^n_k/\mathbb{P}^{n-1}_k]_{\mathbb{A}^1} \to \mathrm{GW}(k)$$

Here $[-,-]_{\mathbb{A}^1}$ denotes the $\mathbb{A}^1$ homotopy classes and $\mathrm{GW}(k)$ the Grothendieck-Witt ring of k.

We will recall the definition of the Grothendieck-Witt ring in the next lecture and treat it as a black box for now. Now we simply copy the definitions from Definition 1.8 and adapt them to the algebraic/motivic setting. The following was first done in [KW21].

Definition 1.15. Let $p\colon V \to X$ be an algebraic vector bundle of rank n over a smooth n-dimensional k-scheme X.

- Let T_X be the tangent bundle of X. A *relative orientation* of $p\colon V \to X$ consists of the data of a line bundle $\mathcal{L} \to X$ and an isomorphism of line bundles $\rho\colon \mathcal{H}om(\det T_X, \det V) \cong \mathcal{L}^{\otimes 2}$. A vector bundle $p\colon V \to X$ is *relatively orientable* if a relative orientation exists.
- Let $x \in X$ be a closed point and let U be a Zariski open neighborhood of x. An étale map $\psi\colon U \to \mathbb{A}^n_k$ which induces an isomorphism of residue fields at x is called *Nisnevich coordinates*. By [KW21, Lemma 19] Nisnevich coordinates always exist whenever $\dim X = n \geq 1$.
- Assume $p\colon V \to X$ is relatively oriented with relative orientation ρ. Let $x \in X$ be a closed point and let $\psi\colon U \to \mathbb{A}^n_k$ be Nisnevich coordinates of x such that there is a local trivialization of $V|_U$. We say that Nisnevich coordinates around x and a trivialization $V|_U \cong U \times \mathbb{A}^n$ are *compatible* with ρ if the distinguished element of $\mathcal{H}om(\det T_X, \det V)$ taking the distinguished basis of $\det T_X|_U$ to $\det V|_U$ is sent to a square by ρ, that is the image of distinguished section can be written as $l \otimes l$ where l is a section of $\mathcal{L}$.
- Assume $p\colon V \to X$ is relatively oriented by ρ and $\sigma\colon X \to V$ is a section. Let x be an isolated zero of σ. Then locally around x choose Nisnevich coordinates and a trivialization of V compatible with ρ. Nisnevich coordinates allow to express the section σ locally around x as a map $\sigma\colon \mathbb{A}^n_k \to \mathbb{A}^n_k$ (for more details see [KW21, §4]) and the local index $\operatorname{ind}_x \sigma$ is the "local $\mathbb{A}^1$-degree" of $\sigma\colon \mathbb{A}^n_k \to \mathbb{A}^n_k$ at x.

Remark 1.16. *We will define the local $\mathbb{A}^1$-degree of a map $f\colon \mathbb{A}^n_k \to \mathbb{A}^n_k$ at a zero in Definition 2.12 and provide a formula for computing it in Proposition 2.13.*

Example 1.17. For both $\mathbb{P}^n_k$ and $\operatorname{Gr}(2,4)$ for any closed point we can find a Zariski local neighborhood isormophic to $\mathbb{A}^n_k$. So here it is straightforward to write down Nisnevich coordinates and express a section σ locally as $\sigma\colon \mathbb{A}^n_k \to \mathbb{A}^n_k$.

Theorem 1.18 (Kass-Wickelgren, Bachmann-Wickelgren)**.** *Suppose $p\colon V \to X$ is an algebraic vector bundle of rank n over a smooth proper n-dimensional k-scheme X relatively oriented by ρ and let σ be a section with only isolated zeros. Then the* Poincaré Hopf Euler number

$$n^{\mathbb{A}^1}(V,\rho) := \sum_{\text{zeros of } \sigma} \operatorname{ind}_x \sigma$$

is a well-defined element of $\operatorname{GW}(k)$. *In particular, this means that $n^{\mathbb{A}^1}(V,\rho)$ is independent of the choice of section.*

Remark 1.19. *We will often omit the relative orientation ρ in $n^{\mathbb{A}^1}(V,\rho)$.*

2. Formulas for the $\mathbb{A}^1$-degree

Content of the lecture

- Definition $\operatorname{GW}(k)$, rank and signature, first examples, generators and relations, the trace form
  Literature: [Lam05]
- The local $\mathbb{A}^1$-degree and formulas for it
  Literature: [Hat02, KW19, EL77, Khi77, BBM+20, BMP23]
- Back to the examples of Bézout and lines on a smooth cubic surface [KW21, McK21]

We start with the definition of the Grothendieck-Witt ring of a field k. For this recall that a set M equipped with a binary operation $\circ\colon M \times M \to M$ is a *monoid* if

- for $a, b, c \in M$ you have $(a \circ b) \circ c = a \circ (b \circ c)$ (associativity)
- there exists $e \in S$ such that $e \circ a = a \circ e = a$ for all $a \in M$ (identity element)

A monoid $(M, \circ)$ is commutative, if the binary operation is commutative, i.e. $a \circ b = b \circ a$ for all $a, b \in M$. A *monoid homomorphism* $f\colon (M, \circ_M) \to (N, \circ_N)$ is a map $f\colon M \to N$ such that

$f(a \circ_M b) = f(a) \circ_N f(b)$ for all $a, b \in M$ and $f(e_M) = e_N$ (these are the identity elements of the respective monoids).

If you also had inverses, this would be a group which is abelian if the monoid is commutative. Groups are quite well understood. For example, finitely generated abelian groups are fully classified. Furthermore, doing computations in groups works much better than in monoids, since you have inverses. So in many situations you want to turn a commutative monoid into an abelian group, which can be done with the following construction.

Definition 2.1. Let $(M, \circ)$ be a commutative monoid. Its *Grothendieck group* or *group completion* is an abelian group $K_0(M)$ together with a monoid homomorphism $i\colon M \to K_0(M)$ which satisfies the following universal property. For any abelian group A and monoid homomorphism $f\colon M \to A$, there exists a unique group homomorphism $g\colon K_0(M) \to A$ such that $f = g \circ i$.

$$\begin{array}{ccc} M & \xrightarrow{\ f\ } & A \\ \big\downarrow{\scriptstyle i} & \nearrow{\scriptstyle \exists! g} & \\ K_0(M) & & \end{array}$$

You can easily show that $K_0(M)$ exists by constructing it. For example, you can find two constructions on Wikipedia.

For the whole section let k be a field. A *symmetric bilinear form* over k is a bilinear map

$$b\colon V \times V \to k$$

where V is a k-vector space such that $b(x, y) = b(y, x)$ for all $x, y \in V$. It is *non-degenerate* if $V \to \operatorname{Hom}_k(V, k)$ given by $(v \mapsto (x \mapsto b(v, x))$ is an isomorphism. Two non-degenerate symmetric bilinear forms $b_1\colon V_1 \times V_1 \to k$ and $b_2\colon V_2 \times V_2 \to k$ are *isometric* if there is a linear isomorphism $\varphi\colon V_1 \to V_2$ such that $b_2(\varphi(x), \varphi(y)) = b_1(x, y)$ for all $x, y \in V_1$. This defines an equivalence relation. Consider the set

$$M := \{\text{non-degenerate symmetric bilinear forms over } k\}/\text{isometry}$$

This can be given a monoid structure $(M, \oplus)$ with addition the direct sum: Let $b_1\colon V_1 \times V_1 \to k$ and $b_2\colon V_2 \times V_2 \to k$ be two non-degenerate symmetric bilinear forms, then $b_1 \oplus b_2\colon (V_1 \oplus V_2) \times (V_1 \oplus V_2) \to k$ is also a non-degenerate symmetric bilinear form. You can easily check that this operation is still well defined when you pass to isometry classes.

Remark 2.2. *When* $\operatorname{char} k \neq 2$ *a non-degenerate symmetric bilinear form* $b\colon V \times V \to k$ *defined a non-degenerate quadratic form by*

$$q\colon V \to k,\ q(x) = b(x, x)$$

and given a non-degenerate quadratic form $q\colon V \to k$ *one gets a non-degenerate symmetric bilinear form*

$$b\colon V \times V \to k,\ b(x, y) = \frac{1}{2}(q(x + y) - q(x) - q(y)).$$

So equivalently we can also think of elements of $\mathrm{GW}(k)$ *as isometry classes of non-degenerate quadratic forms.*

Definition 2.3. The *Grothendieck-Witt group* $\mathrm{GW}(k)$ *of* k is the group completion of the monoid of isometry classes of non-degenerate symmetric bilinear forms over k, that is

$$\mathrm{GW}(k) = K_0(M, \oplus).$$

We can define a second binary operation $\otimes$ on M by taking the tensor product. This turns $\mathrm{GW}(k)$ into a ring.

There is a very nice presentation of $\mathrm{GW}(k)$ which is motivated by the following observation. When $\operatorname{char} k \neq 2$ any non-degenerate symmetric bilinear form $b\colon V \times V \to k$ over k can be diagonalized, that is one can find a basis for V such that the Gram matrix of b is diagonal with entries $a_1, \ldots, a_n \in k^\times$ (here $n = \dim_k V$). In particular, the isometry class of b equals the isometry class of the direct sum of the non-degenerate symmetric bilinear forms $k \times k \to k$ defined by $(x, y) \mapsto a_i xy$ for $i = 1, \dots n$. It follows that $\mathrm{GW}(k)$ is generated by the isometry classes of non-degenerate symmetric bilinear forms of the form $k \times k \to k$ defined by $(x, y) \mapsto axy$ for $a \in k^\times$. We denote these generators by $\langle a \rangle$. The following proposition also holds for $\operatorname{char} k = 2$.

Proposition 2.4. $\mathrm{GW}(k)$ *is generated by symbols* $\langle a \rangle$ *for* $a \in k^\times$ *subject to the relations*

(1) $\langle a \rangle = \langle ab^2 \rangle$ *for* $a, b \in k^\times$,
(2) $\langle a \rangle + \langle b \rangle = \langle a + b \rangle + \langle ab(a + b) \rangle$ *for* $a, b, a + b \in k^\times$,
(3) $\langle a \rangle \langle b \rangle = \langle ab \rangle$ *for* $a, b \in k^\times$.

Exercise 2.5. *Show that it holds that*

$$\langle a \rangle + \langle -a \rangle = \langle 1 \rangle + \langle -1 \rangle$$

for all $a \in k^\times$.

Definition 2.6. The class $h := \langle 1 \rangle + \langle -1 \rangle \in \mathrm{GW}(k)$ is called the *hyperbolic form*.

Definition 2.7. The *Witt ring* $\mathrm{W}(k)$ is defined by

$$\mathrm{W}(k) := \frac{\mathrm{GW}(k)}{(h)} = \frac{\mathrm{GW}(k)}{\mathbb{Z} \cdot h}$$

that is we quotient by the ideal generated by h which is indeed just all integer multiples $\mathbb{Z} \cdot h$ of h by Exercise 2.5.

Let $b \colon V \times V \to k$ be a non-degenerate symmetric bilinear form over k. Then the *rank* of b is defined to be $\mathrm{rk}(b) := \dim_k V$. Clearly, isometric non-degenerate symmetric bilinear forms have the same rank and we get a monoid homomorphism $\mathrm{rk} \colon M \to \mathbb{Z}$ which extends uniquely to $\mathrm{rk} \colon \mathrm{GW}(k) \to \mathbb{Z}$ by the universal property of the Grothendieck group.

Remark 2.8. *An element of* $\mathrm{GW}(k)$ *with* $\operatorname{char} k \neq 2$ *is completely determined by its rank and its image in* $\mathrm{W}(k)$.

Example 2.9. Let k be algebraically closed field. Then the rank is an isomorphism $\mathrm{rk} \colon \mathrm{GW}(k) \xrightarrow{\cong} \mathbb{Z}$.

Assume $k \subset \mathbb{R}$ and let $b \colon V \times V \to k$ be a non-degenerate symmetric bilinear form. We can again choose a basis of V such that the Gram matrix has only diagonal entries $a_1, \dots, a_n \in k^\times$ with $n = \mathrm{rk}(b)$. The signature $\mathrm{sgn}(b)$ of b equals the number of positive a_i minus the number of negative a_i. It is easy to show that the signature also defines a monoid homomorphism $\mathrm{sgn} \colon M \to \mathbb{Z}$ and thus a group homomorphism $\mathrm{sgn} \colon \mathrm{GW}(k) \to \mathbb{Z}$.

Example 2.10. An element of $\mathrm{GW}(\mathbb{R})$ is completely determined by its rank and its signature.

Suppose L/k is a finite separable field extension and let $b \colon V \times V \to L$ be a non-degenerate symmetric bilinear form over L. Then we can transform it into a non-degenerate symmetric bilinear form over k by viewing V as a k vector space and composing b with the field trace $\mathrm{Tr}_{L/k}$ from algebra

$$\mathrm{Tr}_{L/k}(b) \colon V \times V \xrightarrow{b} L \xrightarrow{\mathrm{Tr}_{L/k}} k$$

One checks that this respects isometry classes and thus defines a homomorphism of Grothendieck-Witt groups

$$\mathrm{Tr}_{L/k} \colon \mathrm{GW}(L) \to \mathrm{GW}(k).$$

Exercise 2.11. *Show that*

$$\mathrm{Tr}_{\mathbb{C}/\mathbb{R}}(\langle 1 \rangle) = h.$$

Now let's go back to the $\mathbb{A}^1$-degree map defined by Morel

$$\deg^{\mathbb{A}^1} \colon [\mathbb{P}^n_k/\mathbb{P}^{n-1}_k, \mathbb{P}^n_k/\mathbb{P}^{n-1}_k]_{\mathbb{A}^1} \to \mathrm{GW}(k)$$

and first explain how this relates to the degree map in algebraic topology. If $k \subset \mathbb{R}$ you can take both $\mathbb{C}$-points and $\mathbb{R}$-points of the motivic n-sphere ${\mathbb{P}^n_k}/{\mathbb{P}^{n-1}_k}$ and get S^{2n} respectively S^n. Actually one has the following commutative diagram relating the $\mathbb{A}^1$-degree with the degree from algebraic topology

$$\begin{array}{ccccc} [S^{2n}, S^{2n}] & \xleftarrow{\mathbb{C}\text{-points}} & [{\mathbb{P}^n_k}/{\mathbb{P}^{n-1}_k}, {\mathbb{P}^n_k}/{\mathbb{P}^{n-1}_k}]_{\mathbb{A}^1} & \xrightarrow{\mathbb{R}\text{-points}} & [S^n, S^n] \\ \big\downarrow{\scriptstyle \deg^{\mathbb{A}^1}} & & \big\downarrow{\scriptstyle \deg^{\mathbb{A}^1}} & & \big\downarrow{\scriptstyle \deg} \\ \mathbb{Z} & \xleftarrow{\quad \mathrm{rk} \quad} & \mathrm{GW}(k) & \xrightarrow{\quad \mathrm{sgn} \quad} & \mathbb{Z} \end{array}$$

Let's go back to algebraic topology for a bit. Let

$$f \colon S^n \to S^n$$

and let y be a point in the target with finitely many preimages. For a preimage $x \in f^{-1}(y)$ we can choose $B(x) \subset S^n$ such that x is the only preimage in this ball and such that the boundary $\partial B(x)$ of $B(x)$ is mapped to the boundary $\partial B(y)$ of $B(y)$. Also choose a small ball $B(y)$ around y. Then the local degree $\deg_x f$ at x is the degree of the map

$$\overline{f} \colon S^n \simeq {B(x)}/{\partial B(x)} \to {B(y)}/{\partial B(y)} \simeq S^n.$$

Caution: When choosing the homotopy equivalences one needs to watch orientations.

For example, you can find in Hatcher [Hat02] that

$$\deg f = \sum_{x \in f^{-1}(y)} \deg_x f$$

When y is a regular value then locally around all preimages x the map f is a homeomorphism and thus $\overline{f}$ is a homeomorphism and $\deg_x f \in \{\pm 1\}$. In particular, one has the following formula for $\deg_x f$ from differential topology. Choose local oriented coordinates around x and y. Then in these coordinates the local degree $\deg_x f$ at x

$$\deg_x f = \operatorname{sign}(\det \operatorname{Jac} f(x)) \tag{2}$$

is the sign of the determinant of the Jacobian of f evaluated at x.

The goal for the rest of this section is to define the local $\mathbb{A}^1$-degree and write down an analogous formula for it. This builds on work of Kass-Wickelgren. To define this we need the following fact in the $\mathbb{A}^1$-homotopy category. Assume $U \subset \mathbb{A}^n_k \subset \mathbb{P}^n_k$ and U is Zariski open in $\mathbb{A}^n_k$. Then in the $\mathbb{A}^1$-homotopy category ${U}/{U \setminus \{x\}} \simeq {\mathbb{P}^n_k}/{\mathbb{P}^n_k \setminus \{x\}}$.

Definition 2.12. Let x be an isolated zero of $f \colon \mathbb{A}^n_k \to \mathbb{A}^n_k$. Find a Zariski neighborhood U of x such that $f^{-1}(0) \cap U = \{x\}$. Then the *local* $\mathbb{A}^1$*-degree of* f *at* x is defined to be

$$\deg_x^{\mathbb{A}^1} f := \deg^{\mathbb{A}^1}\left({\mathbb{P}^n_k}/{\mathbb{P}^{n-1}_k} \to {\mathbb{P}^n_k}/{\mathbb{P}^n_k \setminus \{x\}} \simeq {U}/{U \setminus \{x\}} \xrightarrow{\overline{f}} {\mathbb{A}^n_k}/{\mathbb{A}^n_k \setminus \{0\}} \simeq {\mathbb{P}^n_k}/{\mathbb{P}^{n-1}_k} \right).$$

The following proposition of Kass-Wickelgren generalizes the formula (2) from differential topology [KW19].

Proposition 2.13. *Let* x *be a preimage of* 0 *under* $f \colon \mathbb{A}^n_k \to \mathbb{A}^n_k$ *and assume that the residue field* $\kappa(x)$ *is a separable field extension of* $\kappa(y)$. *Furthermore, assume that* $\det \operatorname{Jac} f(x) \neq 0$. *Then*

$$\deg_x f = \operatorname{Tr}_{\kappa(x)/k}(\langle \det \operatorname{Jac} f(x) \rangle) \in \mathrm{GW}(k).$$

So what if $\det \operatorname{Jac} f(x) = 0$ or $\kappa(x)$ is not a separable field extension of k? There are more general formulas. Let's first consider the case that $\det \operatorname{Jac} f(x) = 0$. For this case, Eisenbud-Levine [EL77] and independently Khimshiashvili [Khi77] found a formula for the local degree in algebraic topology. It was shown by Kass-Wickelgren [KW19] that the same formula works for the local $\mathbb{A}^1$-degree when $\kappa(x) = k$ and then later generalized by Brazelton-Burklund-McKean-Montoro-Opie for when $\kappa(x)/k$ is separable. Finally, in [BMP23] it is shown that the "multivariate Bézoutian" gives a formula for the local $\mathbb{A}^1$-degree that always works, in particular even when $\kappa(x)/k$ is not separable. All of this builds on work of Scheja-Storch [SS75], who found a canonical way to assign a non-degenerate symmetric bilinear form to a complete intersection. Note that there is now also a Macaulay2 package that can compute local $\mathbb{A}^1$-degrees [BBE+23].

Now let's return to our two examples in enumerative geometry, namely Bézout and lines on a smooth cubic surface.

Example 2.14 (Bézout continued)**.** Let $p\colon V = \mathcal{O}(d_1) \oplus \ldots \oplus \mathcal{O}(d_n) \to \mathbb{P}^n_k$. Then

$$\mathcal{H}om(\det T_{\mathbb{P}^n_k}, \det V) \cong \omega_{\mathbb{P}^n_k} \otimes \mathcal{O}(d_1 + \ldots + d_n) \cong \mathcal{O}(-n - 1 + \sum d_i)$$

and thus $p\colon V \to \mathbb{P}^n_k$ is relatively orientable if and only if $-n - 1 + \sum d_i$ is even. For example, when $n = 1$ then d_1 must be even, and for $n = 2$ one d_i must be even and the other has to be odd. Assume that all intersections of the hypersurfaces $H_i = V(F_i)$ lie in $U = \{x_0 \neq 0\}$. Let $f_i(x_1, \ldots, x_n) = F_i(1, x_1, \ldots, x_n)$. In [McK21] McKean shows that if the hypersurfaces meet transversely at all intersection points, then at $x \in H_1 \cap \ldots \cap H_n$ the local index

$$\operatorname{ind}_x \sigma_{F_1,\ldots,F_n} = \operatorname{Tr}_{\kappa(x)/k}(\langle \det \operatorname{Jac}(f_1, \ldots, f_n)(x)\rangle)$$

and thus

$$n^{\mathbb{A}^1}(V) = \sum_{x \in H_1 \cap \ldots \cap H_n} \operatorname{Tr}_{\kappa(x)/k}(\langle \det \operatorname{Jac}(f_1, \ldots, f_n)(x)\rangle).$$

McKean also shows that

$$n^{\mathbb{A}^1}(V) = \frac{d_1 \cdot \ldots \cdot d_n}{2} h.$$

Let's compute the rank and the signature of this

$$\operatorname{rk}\left(\frac{d_1 \cdot \ldots \cdot d_n}{2} h\right) = d_1 \cdot \ldots \cdot d_n$$

which is the answer over $k = \mathbb{C}$ and

$$\operatorname{sgn}\left(\frac{d_1 \cdot \ldots \cdot d_n}{2} h\right) = 0$$

which is the real signed count of intersection points.

Example 2.15 (Lines on a cubic surface continued)**.** Kass-Wickelgren [KW21] show that $p\colon V = \operatorname{Sym}^3 \mathcal{S}^* \to \operatorname{Gr}(2,4)$ is relatively orientable and that

$$n^{\mathbb{A}^1}(V) = \sum_{\ell \subset \{f=0\}} \operatorname{ind}_{[\ell]} \sigma_f = 15\langle 1\rangle + 12\langle -1\rangle.$$

Again let's check whether this also yields the complex and real counts when taking the rank and the signature.

$$\operatorname{rk}(15\langle 1\rangle + 12\langle -1\rangle) = 27$$

and

$$\operatorname{sgn}(15\langle 1\rangle + 12\langle -1\rangle) = 3.$$

Kass-Wickelgren also provide an intrinsic way to assign a class $\operatorname{Type}(\ell) \in \operatorname{GW}(\kappa([\ell]))$ to a line $\ell \subset \{f = 0\}$ on the cubic surface such that

$$\operatorname{ind}_{[\ell]} \sigma_f = \operatorname{Tr}_{\kappa([\ell])/k}(\operatorname{Type}(\ell))$$

which generalizes Segre's work over $\mathbb{R}$. Namely, assume that ℓ is defined over k and consider

$$\begin{aligned} G\colon \ell \cong \mathbb{P}^1_k &\to \text{2-planes in } \mathbb{P}^3_k \text{ containing } \ell \cong \mathbb{P}^1_k \\ p &\mapsto T_p Y \end{aligned}$$

where $Y = \{f = 0\} \subset \mathbb{P}^3_k$ is the cubic surface. This turns out to be a degree 2 map and thus there is a non-trivial involution $i\colon \ell \to \ell$ such that $G \circ i = G$ which sends a point to the other point with the same tangent space. The fixed points of this involution are defined over $k(\sqrt{\alpha})$ for some $\alpha \in k^\times$. Kass-Wickelgren show that $\operatorname{Type}(\ell) = \langle \alpha \rangle \in \operatorname{GW}(k)$.

Exercise 2.16. *Convince yourself that for $k = \mathbb{R}$ Kass-Wickelgren's* $\text{Type}(\ell)$ *generalizes Segre's type described in Example 1.11. That is, show that if you follow the normal vector pointing out of the cubic surface along the real line and you make a full turn, then* $\text{Type}(\ell) = \langle -1 \rangle$ *and if you do not make a full turn, then* $\text{Type}(\ell) = \langle 1 \rangle \in \text{GW}(\mathbb{R})$.

Remark 2.17. *In general it is a very hard problem to find an intrinsic description of the local index. A very nice survey of what is known is given in [McK22]. There are also many open problems listed. The definition of the type of a line on a smooth cubic surface can be generalized to lines on degree $2n-3$ hypersurfaces in $\mathbb{P}^n$ by [Pau22] and [EMP24].*

3. Plane tropical curves

Content of the lecture:

- the field of Puiseux series over k, tropical semiring, tropical polynomials and tropical vanishing locus, plane tropical curves, dual subdivision
  Literature: [BS14, MR, MS15]
- Tropical Bézout for curves
  Literature: [BS14, JPP22]

Often one can translate problems inin algebraic geometry to problems in tropical geometry where they can be solved using combinatorics. The great breakthrough in the use of tropical geometry in enumerative geometry was Milkalkin's tropical correspondence theorem, which allows one to translate the count of plane algebraic curves through a given number of points into the count of tropical curves through the same number of points [Mik05] and which we will return to in the last lecture.

In this lecture we explain how to go from algebraic plane curves to tropical plane curves. To do this, we consider algebraic curves over the field of *Puiseux series*, which we define next.

Definition 3.1. Let k be a field. Then the field of Puiseux series over k is

$$k\{\{t\}\} := \bigcup_{n\geq 0} k((t^{\frac{1}{n}})) = \{a_0t^{q_0}+a_1t^{q_1}+\ldots :\ a_i \in k, q_i \in \mathbb{Q} \text{ s.t. } \exists n \in \mathbb{Z}_{>0} \text{ s.t. } q_i \cdot n \in \mathbb{Z} \text{ for all } i\}.$$

Example 3.2. When $\operatorname{char} k = 0$, then $\overline{k((t))} = \overline{k}\{\{t\}\}$.

Example 3.3. Suppe k has characteristic p and is algebraically closed. Consider the following Artin-Schreier polynomial $z^p - z - t^{-1} \in k\{\{t\}\}[z]$. Show that

$$t^{-\frac{1}{p}} + t^{-\frac{1}{p^2}} + t^{-\frac{1}{p^3}} + \ldots$$

is a zero of this polynomial and conclude that $k\{\{t\}\}$ is not algebraically closed.

There is a *valuation* on $k\{\{t\}\}$

$$\text{val}\colon k\{\{t\}\} \to \mathbb{Q} \cup \{\infty\}$$

defined by $\text{val}(a_0t^{q_0} + a_1t^{q_1} + \ldots) = q_0$ and $\text{val}(0) = \infty$. This satisfies

(1) $\text{val}(a(t)) = \infty \Leftrightarrow a(t) = 0$
(2) $\text{val}(a(t) + b(t)) \geq \min\{\text{val}(a(t)), \text{val}(b(t))\}$
(3) $\text{val}(a(t) \cdot b(t)) = \text{val}(a(t)) + \text{val}(b(t))$.

Let's first assume that k is of characteristic 0 and algebraically closed. Then $k\{\{t\}\}$ is also algebraically closed and of characteristic 0. Let's try to find points in the zero locus of a polynomial $F \in k\{\{t\}\}[z_1, z_2]$.

Let's assume that $\deg F = 1$. So we can write

$$F(z_1, z_2) = \widetilde{a}(t) + \widetilde{b}(t)z_1 + \widetilde{c}(t)z_2$$

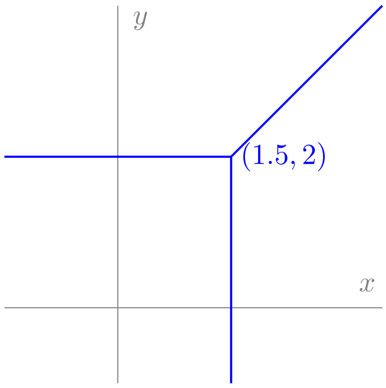


FIGURE 2. $(x, y) \in \mathbb{Q}^2$ for which $F(\widetilde{p}_1, \widetilde{p}_2) = 0$ can be solved with $x = -\operatorname{val}(\widetilde{p}_1)$ and $y = -\operatorname{val}(\widetilde{p}_2)$ in Example 3.4.

with

$$\widetilde{a}(t) = at^{\operatorname{val}(\widetilde{a})} + h.o.t \in k\{\{t\}\}$$
$$\widetilde{b}(t) = bt^{\operatorname{val}(\widetilde{b})} + h.o.t. \in k\{\{t\}\}$$
$$\widetilde{c}(t) = ct^{\operatorname{val}(\widetilde{c})} + h.o.t. \in k\{\{t\}\}$$

here $h.o.t.$ is short for higher order terms in t. Next we want to describe the zeros $(\widetilde{p}_1(t), \widetilde{p}_2(t)) \in k\{\{t\}\}^2$ with

$$\widetilde{p}_1(t) = p_1 t^{\operatorname{val}(\widetilde{p}_1)} + h.o.t.$$
$$\widetilde{p}_2(t) = p_2 t^{\operatorname{val}(\widetilde{p}_2)} + h.o.t.$$

In order for

$$F(\widetilde{p}_1(t), \widetilde{p}_2(t)) = (at^{\operatorname{val}(\widetilde{a})} + h.o.t.) + (bp_1 t^{\operatorname{val}(\widetilde{b})+\operatorname{val}(\widetilde{p}_1)} + h.o.t.) + (cp_2 t^{\operatorname{val}(\widetilde{c})+\operatorname{val}(\widetilde{p}_2)} + h.o.t.) = 0$$

it is necessary that the minimum of the three exponents of the three summands

$$\{\operatorname{val}(\widetilde{a}), \operatorname{val}(\widetilde{b}) + \operatorname{val}(\widetilde{p}_1), \operatorname{val}(\widetilde{c}) + \operatorname{val}(\widetilde{p}_2)\}$$

is attained at least twice or equivalently the maximum in

$$\{-\operatorname{val}(\widetilde{a}), -\operatorname{val}(\widetilde{b}) - \operatorname{val}(\widetilde{p}_1), -\operatorname{val}(\widetilde{c}) - \operatorname{val}(\widetilde{p}_2)\}$$

is attained at least twice. Only then one gets a linear equation in p_1 and p_2.

Example 3.4. Let's look at an example and assume $\operatorname{val}(\widetilde{a}) = 2$, $\operatorname{val}(\widetilde{b}) = -\frac{1}{2}$ and $\operatorname{val}(\widetilde{c}) = 0$ and plot for which $x = -\operatorname{val}(\widetilde{p}_1)$ and $y = -\operatorname{val}(\widetilde{p}_2)$ this can be solved in Figure 2. This is actually our first example of a tropical plane curve.

Now let's do this for polynomials $F(z_1, z_2) \in k\{\{t\}\}[z_1, z_2]$ of higher degree. Let $F(z_1, z_2) = \sum \widetilde{a}_{ij}(t) z_1^i z_2^j \in k\{\{t\}\}[z_1, z_2]$ of degree $d \geq 1$ with $\widetilde{a}_{ij}(t) = a_{ij} t^{\operatorname{val}(\widetilde{a}_{ij})} + h.o.t. \in k\{\{t\}\}$. Then $F(z_1, z_2)$ can only have a zero $(\widetilde{p}_1, \widetilde{p}_2) \in k\{\{t\}\}^2$ if

$$\max_{ij}\{i \cdot x + j \cdot y - \operatorname{val}(\widetilde{a}_{ij})\} \tag{3}$$

with $x = -\operatorname{val}(\widetilde{p}_1)$ and $y = -\operatorname{val}(\widetilde{p}_2)$ is attained at least twice.

As before, set $x = -\operatorname{val}(\widetilde{p}_1)$ and $y = -\operatorname{val}(\widetilde{p}_2)$ for $\widetilde{p}_1(t) = p_1 t^{\operatorname{val}(\widetilde{p}_1)} + h.o.t. \in k\{\{t\}\}$ and $\widetilde{p}_2(t) = p_2 t^{\operatorname{val}(\widetilde{p}_2)} + h.o.t. \in k\{\{t\}\}$. Then it follows from the properties of the valuation (1)-(3) listed above that

(1) $x = -\infty \Leftrightarrow \widetilde{p}_1(t) = 0$ (and the same for y and $\widetilde{p}_2$)
(2) $-\operatorname{val}(\widetilde{p}_1(t) + \widetilde{p}_2(t)) \leq \max\{x, y\}$
(3) $-\operatorname{val}(\widetilde{p}_1(t) \cdot \widetilde{p}_2(t)) = x + y$

This motivates the following definition of the tropical semi-ring.

Definition 3.5. The *tropical semi-ring* $(\mathbb{T}, \oplus, \otimes)$ is defined by

- $\mathbb{T} = \mathbb{R} \cup \{-\infty\}$
- $x \oplus y = \max(x, y)$
- $a \odot b = a + b$

This is not a ring/field since there are no inverses to tropical addition $\oplus$. However, $-\infty$ is the neutral element for tropical addition $\oplus$.

Definition 3.6. A *tropical polynomial* in two variables x, y is of the form

$$f = \bigoplus a_{ij} \odot x^{\odot i} y^{\odot j} = \max\{a_{ij} + ix + jy\}$$

where $x^{\odot i} = x \odot \ldots \odot x$ with i tropical factors and $y^{\odot j} = y \odot \ldots \odot y$ with j tropical factors. with only finitely many a_{ij} not equal to $-\infty$, that is the neutral element for tropical addition.

The *tropical vanishing locus* of a tropical polynomial f is given by

$$V^{\text{trop}}(f) = \{p \in \mathbb{R}^2 : \text{the maximum is attained at least twice at } p\}$$

This might look familiar: Recall that given a polonomial $F(z_1, z_2) \in k\{\{t\}\}[z_1, z_2]$, this describes a necessary condition for a zero to exist in terms of the valuation (3). In fact, one can turn a polynomial

$$F(x, y) = \sum \widetilde{a}_{ij}(t) z_1^i z_2^j \in k\{\{t\}\}[z_1, z_2]$$

into a tropical polynomial

$$F^{\text{trop}}(x, y) = \bigoplus - \operatorname{val}(\widetilde{a}_{ij}(t)) \odot x^{\odot i} y^{\odot j}.$$

F^{trop} is called the *tropicalization* of F. The following Theorem implies that the necessary condition for $(\widetilde{p}_1(t), \widetilde{p}_2(t)) \in k\{\{t\}\}^2$ to be in the vanishing locus of F is actually sufficient.

Theorem 3.7 (Kapranov)**.** *Let k be an algebraically closed field of characteristic 0. Then $V^{\text{trop}}(F^{\text{trop}})$ equals the closure of*

$$\{(-\operatorname{val}(\widetilde{p}_1(t)), -\operatorname{val}(\widetilde{p}_2(t))) \in \mathbb{Q}^2 : F(\widetilde{p}_1(t), \widetilde{p}_2(t)) = 0\}$$

in $\mathbb{R}^2$.

We are ready to define plane tropical curves.

Definition 3.8. A *plane tropical curve* Γ is an embedded weighted graph in $\mathbb{R}^2$ given by the tropical vanishing locus of a tropical polynomial f. The weights on the edges are defined as follows. Let e be an edge, then its weight $w(e)$ is

$$w(e) := \max\{\gcd(|i-k|, |j-l|) : \ ((i,j),(k,l)) \in M_e\}$$

with $M_e = \{((i,j),(k,l)) \in \mathbb{Z}^2_{>0} : \ \forall (p,q) \in e, f(p,q) = a_{ij} + ip + jq = a_{kl} + kp + lq\}$ that is M_e ranges over pairs of tuples where the maximum is attained.

Remark 3.9. *One only labels edges of weight > 1. When there is no label, it means that the weight of this edge is 1 (see Figure 3).*

Definition 3.10. Let C be a plane algebraic curve over $k\{\{t\}\}$ defined by $F \in k\{\{t\}\}[z_1, z_2]$ and let Γ be the tropical curve defined by F^{trop}. We say C *tropicalizes* to Γ. We also say a closed point $(\widetilde{p}_1(t), \widetilde{p}_2(t)) \in k\{\{t\}\}^2$ *tropicalizes* to $(-\operatorname{val}(\widetilde{p}_1(t)), -\operatorname{val}(\widetilde{p}_2(t))) \in \mathbb{R}^2$.

Example 3.11. Consider the following tropical polynomial

$$f = 0 \oplus x \oplus y \oplus (-1) \odot x \odot y \oplus (-2) \odot x^{\odot 2} \oplus (-2) \odot y^{\odot 2} = \max\{0, x, y, x+y-1, 2x-2, 2y-2\}.$$

Figure 3 shows the tropical curve defined by f. In the figure it is also indicated in red which piecewise linear functions out of $\{0, x, y, x+y-1, 2x-2, 2y-2\}$ is maximal in which segment.

For this curve all edges are of weight one.

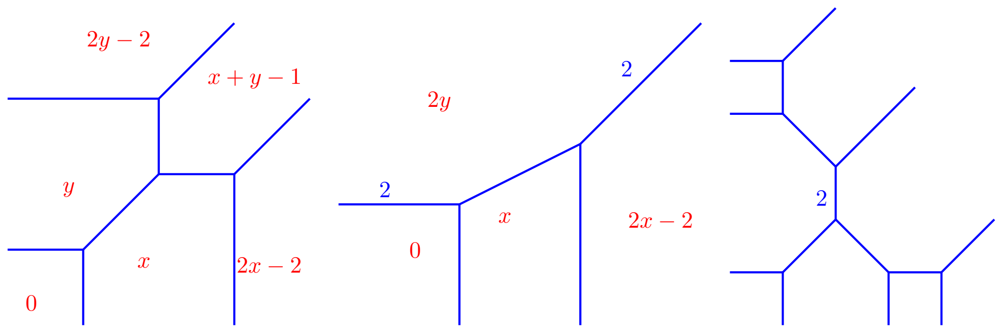


FIGURE 3. Examples of tropical curves

Example 3.12. The following example is inspired by [BS14, Figure 3]. It is the tropical curve defined by the tropical polynomial

$$f = 0 \oplus x \oplus y \oplus y^{\odot 2} \oplus (-2) \odot x^{\odot 2}$$

and shown in the middle in Figure 3. Here, we have edges of weight bigger than 1. For example take the edge of weight 2 on the left, that is the line $y = 0$. Here, the maximum is attained by 0, y and $2y$. The corresponding tropical monomials are 0, y and $y^{\odot 2}$. So for this edge, let's call it e, we have $M_e = \{(0,0),(0,1),(0,2)\}$ and thus $w(e) = \max\{\gcd(|0-0|,|1-0|), \gcd(|0-0|,|2-0|), \gcd(|0-0|,|2-1|)\} = 2$.

For the right edge, let's call it e', the maximum is attained by $2x-2$ and $2y$ and we have $M_{e'} = \{(2,0),(0,2)\}$ and thus $w(e') = \gcd(2,2) = 2$.

Example 3.13. The right picture in Figure 3 is a tropical curve defined by a degree 3 tropical polynomial.

Definition 3.14. Let Γ be a plane tropical curve defined by a tropical polynomial $f = \bigoplus a_{ij} \odot x^{\odot i} \odot y^{\odot j}$. The *Newton polygon* $\mathrm{NP}(f)$ of f is the convex hull

$$\mathrm{Conv}\{(i,j) : \ a_{ij} \neq -\infty\} \subset \mathbb{R}^2$$

The Newton polygon is a lattice polygon in $\mathbb{R}^2$ meaning that the vertices are in $\mathbb{Z}^2$. To a tropical polynomial $f = \bigoplus a_{ij} \odot x^{\odot i} \odot y^{\odot j}$ one can assign a lattice subdivision of its Newton polygon called the *dual subdivision* $\mathrm{DS}(f)$ in the following way. One projects the edges of the upper faces of

$$\mathrm{Conv}(\{(i,j,a_{ij}) : a_{ij} \neq -\infty\}) \subset \mathbb{R}^3$$

to $\mathbb{R}^2$ via the projection to the first two coordinates.

One can show that there is the following one-to-one correspondence

Tropical curve Γ defined by f	**Dual subdivision $\mathrm{DS}(f)$**
vertex	connected component of $\mathrm{NP}(f) \setminus \mathrm{DS}(f)$
edge of weight w	edge of lattice length w
connected component of $\mathbb{R}^2 \setminus \Gamma$	vertex

Moreover dual edges are orthogonal and inclusions are inverted.

Example 3.15. Figure 4 shows the dual subdivisions of the tropical curves in Figure 3.

Exercise 3.16. *Use the dual subdivision to show that plane tropical curves satisfy the* balancing condition*: At every vertex $v \in \Gamma$ and every edge e with vertex e, let $u_e \in \mathbb{Z}^2$ the vector pointing away from v in direction e with entries coprime (the u_e are drawn as arrows in Figure 5). Then*

$$\sum_{v \in e} w(e) u_e = 0.$$

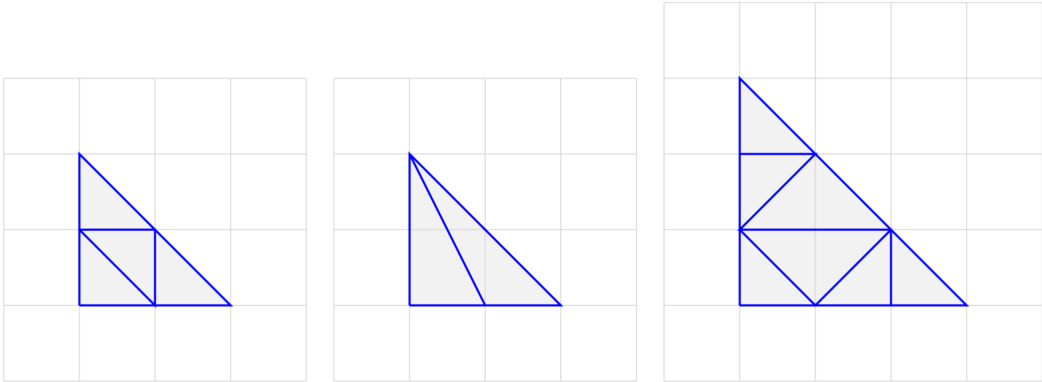


FIGURE 4. Dual subdivisions

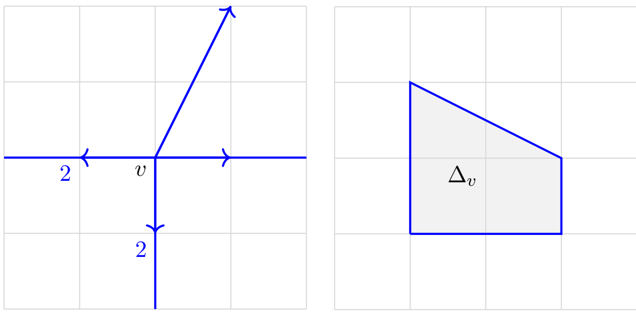


FIGURE 5. Balancing condition

We conclude this lecture by explaining an application of tropical geometry to enumerative geometry by showing Bézout's theorem for tropical curves. This will imply Bézout's theorem for curves over $k\{\{t\}\}$ where k is algebraically closed and $\operatorname{char} k = 0$. Let's start with two algebraic curves C_1 and C_2 over $k\{\{t\}\}$ defined by polyonimals $F_1, F_2 \in k\{\{t\}\}[z_1, z_2]$. Let Γ_1 and Γ_2 be the two tropical curves defined by the tropical polynomials F_1^{trop} and F_2^{trop}, respectively. Assume that $(\widetilde{p}_1(t), \widetilde{p}_2(t)) \in C_1 \cap C_2$ is an intersection point. Then the tropical curves Γ_1 and Γ_2 intersect in the tropicalization $p := (-\operatorname{val}(\widetilde{p}_1(t)), -\operatorname{val}(\widetilde{p}_2(t))) \in \mathbb{R}^2$ of $(\widetilde{p}_1(t), \widetilde{p}_2(t))$.

Definition 3.17. Γ_1 and Γ_2 intersect *tropically transversally* at p if

- p is an *isolated intersection point*, that is there is a small open ball around p such that p is the only intersection point of Γ_1 and Γ_2,
- p is not a vertex of Γ_1 or Γ_2.

The union $\Gamma_1 \cup \Gamma_2$ is the tropical curve defined by the tropical polynomial $(F_1 \cdot F_2)^{\text{trop}} = F_1^{\text{trop}} \odot F_2^{\text{trop}}$ and $p \in \Gamma_1 \cup \Gamma_2$ is a 4-valent vertex of $\Gamma_1 \cup \Gamma_2$ in case Γ_1 and Γ_2 intersect tropically transversally at p. Thus dual to this 4-valent vertex is a quadrilateral Δ_p. In fact Δ_p is a parallelogram since two opposite edges in the dual quadrilateral must be parallel since both have to be orthogonal to the same edge of one of the curves, as illustrated in Figure 6.

Lemma 3.18. *Let $p \in \Gamma_1 \cap \Gamma_2$ be an intersection point and assume that Γ_1 and Γ_2 intersect tropically transversally at p. Then the number of intersection points $(\widetilde{p}_1(t), \widetilde{p}_2(t)) \in C_1 \cap C_2$ counted with multiplicity such that $(-\operatorname{val}(\widetilde{p}_1(t)), -\operatorname{val}(\widetilde{p}_2(t))) = p \in \mathbb{R}^2$ equals* $\operatorname{Area}(\Delta_p)$.

This motivates the following definition.

Definition 3.19. Let Γ_1 and Γ_2 be two tropical curves and let $p \in \Gamma_1 \cap \Gamma_2$. Then the *intersection multiplicity* of Γ_1 and Γ_2 at p is given by

$$\operatorname{mult}_p(\Gamma_1, \Gamma_2) := \operatorname{Area}(\Delta_p)$$

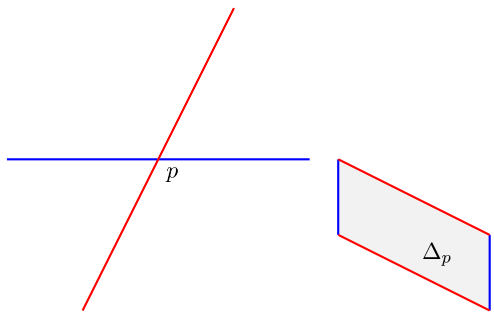


FIGURE 6. An intersection point p of two tropical curves Γ_1 and Γ_2 and its dual parallelogram Δ_p

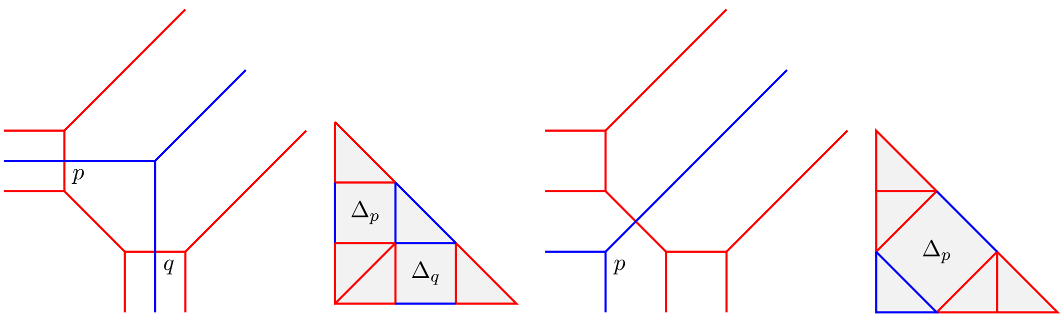


FIGURE 7. A tropical conic and a tropical line intersecting in two points with multiplicity 1 on the left and in one point with multiplicity 2 on the right

where Δ_p is the parallelogram dual to the 4-valent vertex p of the tropical curve $\Gamma_1 \cup \Gamma_2$ in the dual subdivision $\mathrm{DS}(\Gamma_1 \cup \Gamma_2)$.

Example 3.20. Figure 7 shows a tropical line (in blue) intersecting a tropical conic (in red) in two different ways. On the left there are two intersections and both have intersection multiplicity 1 and on the right there is only one intersection point with intersection multiplicity 2.

Let

$$\Delta_d \coloneqq \mathrm{Conv}\{(0,0),(d,0),(0,d)\}.$$

This is the Newton polygon of a (tropical) polyomial $\bigoplus_{0\le i+j\le d} a_{ij} \odot x^{\odot i} \odot y^{\odot j}$ of degree d with coefficients $a_{ij} \neq -\infty$.

Theorem 3.21 (Bézout theorem for tropical curves)**.** *Let Γ_1 and Γ_2 be two tropical curves with Newton polygons Δ_{d_1} respectively Δ_{d_2}. Then*

$$\sum_{p\in\Gamma_1\cap\Gamma_2} \mathrm{mult}_p(\Gamma_1,\Gamma_2) = d_1 \cdot d_2.$$

Proof. The tropical curve $\Gamma_1 \cup \Gamma_2$ has Newton polygon $\Delta_{d_1+d_2}$. The dual subdivision of $\Gamma_1 \cup \Gamma_2$ consists of the dual subdivision of Γ_1, Γ_2 and the parallelograms corresponding to the intersection points as indicated by the different colors in Figure 8. Thus

$$\begin{aligned}\sum_{p\in\Gamma_1\cap\Gamma_2} \mathrm{mult}_p(\Gamma_1,\Gamma_2) &= \mathrm{Area}(\Delta_{d_1+d_2}) - \mathrm{Area}(\Delta_{d_1}) - \mathrm{Area}(\Delta_{d_2}) \\ &= \frac{(d_1+d_2)^2}{2} - \frac{d_1^2}{2} - \frac{d_2^2}{2} = d_1 \cdot d_2\end{aligned}$$

□

Now together with Lemma 3.18 this implies

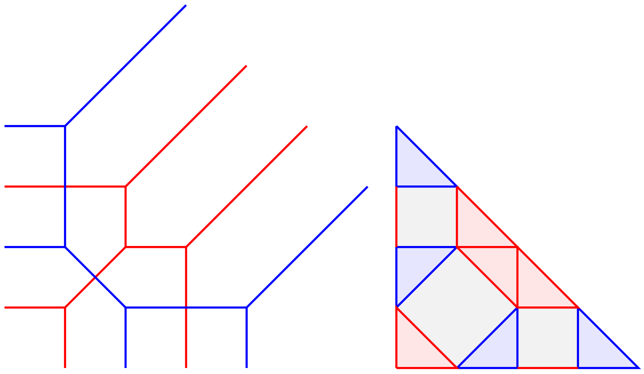


FIGURE 8. Two tropical degree 2 curves intersecting with the dual subdivision of their union.

Corollary 3.22 (Bézout over $k\{\{t\}\}$). *Let C_1 and C_2 be two algebraic curves in $\mathbb{P}^2_{k\{\{t\}\}}$ of degree d_1 respectively d_2. Then*

$$\sum_{\widetilde{p}\in C_1\cap C_2} \operatorname{mult}_{\widetilde{p}}(C_1, C_2) = d_1 \cdot d_2.$$

Remark 3.23. *This also works for tropical curves with Newton polygons other than Δ_d. In this case $\sum_{p\in\Gamma_1\cap\Gamma_2} \operatorname{mult}_p(\Gamma_1, \Gamma_2)$ equals the* mixed volume *of the two Newton polygons. This again implies the analogous algebraic statement over $k\{\{t\}\}$ known as the Bernstein-Kushnerenko theorem. All of this also works in higher dimensions and dimension 1, that is not only for (tropical) curves but also for (tropical) hypersurfaces.*

4. Tropical enumerative geometry

Contents:

- A tropical Bézout theorem for curves in $\mathrm{GW}(k)$
  Literature: [JPP22]
- The count of plane rational degree d curves and tropical correspondence theorems.
  Literature: [Mik05],[Wel05], [Shu06], [KLSW23a], [PP24]

In the last lecture we will bring together the two worlds of tropical geometry and enumerative geometry in $\mathrm{GW}(k)$. To avoid non-separable field extensions and zeros of our polynomials not contained in $\overline{k}\{\{t\}\}$, we assume that the characteristic of k is either 0 or big, that is larger than the degrees of the polynomials.

Let's start with Bézout's theorem for curves. Let $k\{\{t\}\}$ be the field of Puiseux series over k (and k not necessarily algebraically closed anymore and we also allow positive characteristic). Then we have the following Bézout theorem for curves over $k\{\{t\}\}$ (see Example 2.14) which we will reprove using tropical geometry.

Theorem 4.1. *Let C_1 and C_2 be curves in $\mathbb{P}^2_{k\{\{t\}\}}$ of degree d_1 respectively d_2 with $d_1 + d_2$ odd which meet transversally and with all intersection points contained in some $\mathbb{A}^2_{k\{\{t\}\}}$. Then as subsets of $\mathbb{A}^2_{k\{\{t\}\}}$ one can write $C_1 = V(F_1)$ and $C_2 = V(F_2)$ with $F_1, F_2 \in k\{\{t\}\}[z_1, z_2]$. It then holds that*

$$\sum_{\widetilde{p}\in C_1\cap C_2} \operatorname{Tr}_{\kappa(\widetilde{p})/k}\langle \det \operatorname{Jac}(F_1, F_2)(\widetilde{p})\rangle = \frac{d_1 \cdot d_2}{2} h \in \mathrm{GW}(k\{\{t\}\}) \cong \mathrm{GW}(k)$$

where $\kappa(\widetilde{p})$ is the residue field of $\widetilde{p}$.

Let Γ_1 and Γ_2 be the two tropical curves defined by F_1^{trop} and F_2^{trop}. To prove Theorem 4.1 we need an analog of Lemma 3.18 yielding a tropical intersection multiplicity valued in $\mathrm{GW}(k) \cong \mathrm{GW}(k\{\{t\}\})$.

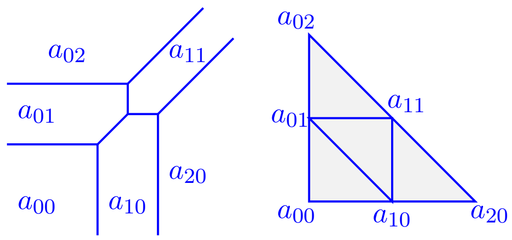


FIGURE 9. An enriched tropical curve and its enriched dual subdivision.

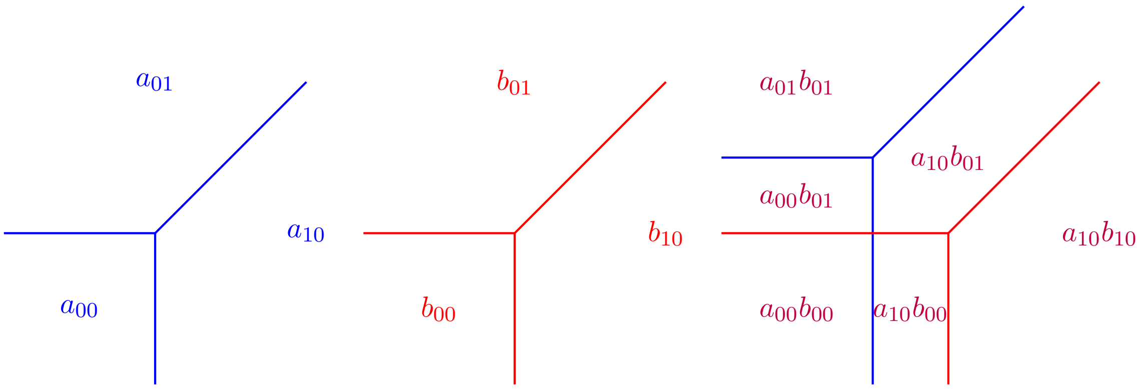


FIGURE 10. The union of two enriched tropical curves.

However, by passing from C_i to Γ_i for $i = 1, 2$ we lose too much information when k is not algebraically closed. We need to "enrich" our tropical curves. The following exercise already indicates that we only need to remember the coefficients of F_1 and F_2 up to squares in $k\{\{t\}\}^\times$.

Exercise 4.2. (1) *There is a bijection*

$$k\{\{t\}\}^\times/(k\{\{t\}\}^\times)^2 \cong k^\times/k^\times, \; a(t) = a_0 t^{q_0} + h.o.t. \mapsto a_0$$

(2) *Conclude that* $\mathrm{GW}(k\{\{t\}\}) \to \mathrm{GW}(k)$ *defined by* $\langle a_0 t^{q_0} + h.o.t.\rangle \mapsto \langle a_0\rangle$ *is an isomorphism by checking that this map respects the relations in the Grothendieck-Witt rings.*

Let $F(z_1, z_2) = \sum \widetilde{a}_{ij}(t) z_1^i z_2^j \in k\{\{t\}\}[z_1, z_2]$. Now recall that the connected components of the complement of a tropical curve correspond to the tropical monomials in the tropical polynomial F^{trop} where the maximum is attained, which correspond to monomials of F. We want to remember the coefficients $\widetilde{a}_{ij}(t) = a_{ij} t^{\mathrm{val}(\widetilde{a}_{ij})} + h.o.t.$ in F up to squares in $k\{\{t\}\}$ which by the above exercise is equal to $a_{ij} \in k^\times/(k^\times)^2$ and we do this by decorating the components of the complement of the associated tropical curve with a_{ij}. Recall that the components of the complement of the tropical curve correspond to the vertices in the dual subdivision, so we also label these as illustrated in Figure 9 which leads to the following definition.

Definition 4.3. An *enriched tropical curve* over k consists of a tropical curve Γ together with $a_{ij} \in k^\times/(k^\times)^2$ for each component of $\mathbb{R}^2 \setminus \Gamma$ or equivalently vertex (i, j) in the dual subdivision of Γ. We call the a_{ij} *coefficients* at the component/vertex.

Definition 4.4. The union of two enriched tropical curves is again an enriched tropical curve, where the coefficient of a component is given by the product of the coefficients of the respective components of the two tropical curves, as shown for the example of the union of two enriched tropical curves in Figure 10.

Definition 4.5 (Tropical $\mathbb{A}^1$-intersection multiplicity)**.** Let $F_1, F_2 \in k\{\{t\}\}[z_1, z_2]$ and let Γ_1 and Γ_2 be the two associated enriched tropical curves. Assume Γ_1 and Γ_2 intersect tropically

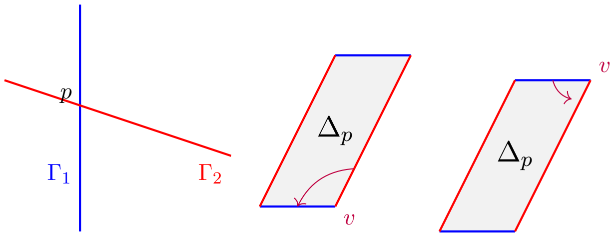


FIGURE 11. For v in the middle $\epsilon_p(v) = -1$ and for v on the right $\epsilon_p(v) = +1$

transversally at p. We define the *tropical* $\mathbb{A}^1$*-intersection multiplicity* of Γ_1 and Γ_2 at p to be

$$\operatorname{mult}_p^{\mathbb{A}^1}(\Gamma_1, \Gamma_2) := \sum_{\widetilde{p}} \operatorname{Tr}_{\kappa(\widetilde{p})/k\{\{t\}\}}(\langle \det \operatorname{Jac}(F_1, F_2)(\widetilde{p})\rangle) \in \mathrm{GW}(k\{\{t\}\})$$

where the sum runs over the common zeros $\widetilde{p}$ of F_1 and F_2, which tropicalize to p. Here, $\kappa(\widetilde{p})$ denotes the residue field of $\widetilde{p}$.

We say that a lattice point $v \in \mathbb{Z}^2$ is *odd* if both entries are odd. The following lemma is the generalization of Lemma 3.18 over arbitrary fields.

Theorem 4.6 (Jaramillo Puentes-Pauli)**.** *Assume that k is a field of characteristic 0 or big. Let Δ_p be the parallelogram in $\mathrm{DS}(\Gamma_1 \cup \Gamma_2)$ dual to $p \in \Gamma_1 \cap \Gamma_2$. Then*

$$\operatorname{mult}_p^{\mathbb{A}^1}(\Gamma_1, \Gamma_2) = \frac{\operatorname{Area}(\Delta_p) - \#\{v \text{ odd vertex of } \Delta_p\}}{2} \cdot h + \sum_{v \text{ odd vertex of } \Delta_p} \langle \epsilon_p(v) a_v \rangle$$

where $a_v \in k^\times/(k^\times)^2$ is the coefficient of the vertex v and

$$\epsilon_p(v) = \begin{cases} +1 & \text{if one starts at an edge dual to an edge of } \Gamma_1 \\ -1 & \text{if one starts at an edge dual to an edge of } \Gamma_2 \end{cases}$$

when walking around v inside of Δ_p anticlockwise as illustrated in Figure 11.

Example 4.7. Let's go back to Example 3.20. Let Γ_1 be the tropical line (in blue) and Γ_2 be the tropical conic (in red) in Figure 7. There is only one odd lattice point inside of Δ_3, namely $(1, 1)$. Let $a_{11} \in k^\times/(k^\times)^2$ be the coefficient of $(1, 1)$. For the intersection on the left, the tropical $\mathbb{A}^1$-intersection multiplicities are as follows

$$\operatorname{mult}_p(\Gamma_1, \Gamma_2) = \langle a_{11} \rangle,\ \operatorname{mult}_q(\Gamma_1, \Gamma_2) = \langle -a_{11} \rangle.$$

To compute the tropical $\mathbb{A}^1$-intersection multiplicity on the right note that Δ_p has no odd vertices (there is one odd lattice point in the interior but this does not count). The area of Δ_p on the right is 2, so using the formula in Theorem 4.6 we get

$$\operatorname{mult}_p^{\mathbb{A}^1}(\Gamma_1, \Gamma_2) = \frac{2}{2} h = h.$$

The proof of Theorem 4.6 is straightforward, yet tedious: One computes $\operatorname{mult}_p^{\mathbb{A}^1}(\Gamma_1, \Gamma_2)$ as defined in Definition 4.5 and identifies it with the combinatorial formula in Theorem 4.6.

Example 4.8. Let's verify Theorem 4.6 for the case of two lines intersection. Let $F_1 = a_1(t)z_1 + b_1(t)z_2 + c_1(t) \in k\{\{t\}\}[z_1, z_2]$ and $F_2 = a_2(t)z_1 + b_2(t)z_2 + c_2(t) \in k\{\{t\}\}[z_1, z_2]$. Then the vertex of the tropical line Γ_1 associated with F_1 lies on the line

$$y = x + \operatorname{val}(b_1) - \operatorname{val}(a_1)$$

and the vertex of the tropical line Γ_2 associated with F_2 lies on the line

$$y = x + \operatorname{val}(b_2) - \operatorname{val}(a_2).$$

Furthermore, the coefficient of $z_1 \cdot z_2$ in $F_1 \cdot F_2$ is $a_1(t)b_2(t) + a_2(t)b_1(t)$. Let $(a_1)_0$, $(a_2)_0$, $(b_1)_0$ and $(b_2)_0$ be the lowest non-vanishing coefficients of $a_1(t)$, $a_2(t)$, $b_1(t)$ and $b_2(t)$, respectively. Then the lowest non-vanishing coefficient of $a_1(t)b_2(t) + a_2(t)b_1(t)$ is

- $(a_1)_0(b_2)_0$ if $\mathrm{val}(a_1) + \mathrm{val}(b_2) < \mathrm{val}(a_2) + \mathrm{val}(b_1)$
- $(a_2)_0(b_1)_0$ if $\mathrm{val}(a_1) + \mathrm{val}(b_2) > \mathrm{val}(a_2) + \mathrm{val}(b_1)$

Note that this agrees with the coefficient of the odd vertex $(1,1)$ of the dual subdivision of $\Gamma_1 \cup \Gamma_2$ in both cases.

Let's compute the enriched intersection multiplicity.

$$\langle \det \mathrm{Jac}(F_1, F_2) \rangle = \langle a_1(t)b_2(t) - a_2(t)b_1(t) \rangle \in \mathrm{GW}(k\{\{t\}\})$$

Under the isomorphism from Exercise 4.2 this agrees with

- $\langle (a_1)_0(b_2)_0 \rangle$ if $\mathrm{val}(a_1) + \mathrm{val}(b_2) < \mathrm{val}(a_2) + \mathrm{val}(b_1)$
- $\langle -(a_2)_0(b_1)_0 \rangle$ if $\mathrm{val}(a_1) + \mathrm{val}(b_2) > \mathrm{val}(a_2) + \mathrm{val}(b_1)$.

Now it is easy to check that this is indeed equal to the combinatorial formula in Theorem 4.6, the minus in the second case is the sign $\epsilon_p(v)$.

Corollary 4.9 (Tropical Bézout for enriched tropical curves)**.** *Assume k is a field of characteristic not equal to 2. Further, assume $d_1 + d_2 \equiv 1 \mod 2$. Then*

$$\sum_{p \in \Gamma_1 \cap \Gamma_2} \mathrm{mult}_p^{\mathbb{A}^1}(\Gamma_1, \Gamma_2) = \frac{d_1 \cdot d_2}{2} \cdot h \in \mathrm{GW}(k).$$

Proof. First, observe that $d_1 + d_2$ odd implies that there are no odd lattice points on $\partial \Delta_{d_1+d_2}$. In other words, all odd lattice points lie in the interior of $\Delta_{d_1+d_2}$. Let v be a lattice point in the interior of $\Delta_{d_1+d_2}$ (for example an odd lattice point). Then, if you make a full turn around this point, observe that you change color exactly when v is a vertex of a parallolgram corresponding to an intersection point and also observe that the order of color change determines the sign $\epsilon_p(v)$. Therefore,

- The number of parallelograms in $\mathrm{DS}(\Gamma_1 \cup \Gamma_2)$ corresponding to an intersection of Γ_1 and Γ_2 with vertex v is even.
- $\#\{p \in \Gamma_1 \cap \Gamma_2 : v \text{ vertex of } \Delta_p, \ \epsilon_p(v) = +1\} = \#\{p \in \Gamma_1 \cap \Gamma_2 : v \text{ vertex of } \Delta_p, \ \epsilon_p(v) = -1\}$.

So in $\mathrm{W}(k) = \frac{\mathrm{GW}(k)}{\mathbb{Z} \cdot h}$ we have

$$\sum_{p \in \Gamma_1 \cap \Gamma_2} \mathrm{mult}_p^{\mathbb{A}^1}(\Gamma_1, \Gamma_2) = \sum_{v \text{ odd}} \frac{\#\{p \in \Gamma_1 \cap \Gamma_2 : v \in \Delta_p\}}{2} (\langle a_v \rangle + \langle -a_v \rangle).$$

Now recall from Exercise 2.5 that $\langle a_v \rangle + \langle -a_v \rangle = h$ and thus in $\mathrm{W}(k)$ we get

$$\sum_{p \in \Gamma_1 \cap \Gamma_2} \mathrm{mult}_p^{\mathbb{A}^1}(\Gamma_1, \Gamma_2) = 0.$$

Recall from Remark 2.8 that when $\mathrm{char}\, k \neq 2$ an element of $\mathrm{GW}(k)$ is uniquely determined by its image in $\mathrm{W}(k)$ and its rank. Thus in $\mathrm{GW}(k)$

$$\sum_{p \in \Gamma_1 \cap \Gamma_2} \mathrm{mult}_p^{\mathbb{A}^1}(\Gamma_1, \Gamma_2) = \frac{\mathrm{rk}\left(\sum_{p \in \Gamma_1 \cap \Gamma_2} \mathrm{mult}_p^{\mathbb{A}^1}(\Gamma_1, \Gamma_2)\right)}{2} h \overset{3.21}{=} \frac{d_1 \cdot d_2}{2} h.$$

□

Example 4.10. Let's continue Example 3.20 and Example 4.7 and check whether we get the correct result. As computed in Example 4.7 for the left picture in Figure 7 we get that

$$\mathrm{mult}_p^{\mathbb{A}^1}(\Gamma_1, \Gamma_2) + \mathrm{mult}_q^{\mathbb{A}^1}(\Gamma_1, \Gamma_2) = \langle a_{11} \rangle + \langle -a_{11} \rangle = h.$$

In the right picture, there is only one intersection point p and we computed in Example 4.7 that $\text{mult}_p^{\mathbb{A}^1}(\Gamma_1, \Gamma_2) = h$. So both sums of $\mathbb{A}^1$-intersection multiplicities agree and they also agree with $\frac{d_1 \cdot d_2}{2} h = \frac{1 \cdot 2}{2} h = h$ as they should by Corollary 4.9.

Now Theorem 4.1 follows directly from Theorem 4.6 and Corollary 4.9.

Remark 4.11. *This also works in other dimensions, that is not only for curves but for hypersurfaces, and other Newton polytopes leading to a quadratically enriched Bernstein-Kushnirenko theorem as shown in* [JPP22].

Remark 4.12. *We can actually deduce McKean's Bézout's theorem (see Example 2.14) over any field of characteristic* 0 *or big from Theorem 4.1. The reason for this is that the Poincaré Hopf Euler number (see Theorem 1.18) of the vector bundle* $\mathcal{O}(d_1) \oplus \mathcal{O}(d_2) \to \mathbb{P}^2_k$ *is sent to the Poincaré Hopf Euler number of* $\mathcal{O}(d_1) \oplus \mathcal{O}(d_2) \to \mathbb{P}^2_{k\{\{t\}\}}$ *under the isomorphism* $\text{GW}(k) \xrightarrow{\cong} \text{GW}(k\{\{t\}\})$ *(this isomorphism is induced by the inclusion* $k \hookrightarrow k\{\{t\}\}$ *and inverse to the map in Exercise 4.2).*

Remark 4.13. *Note that the proof of Corollary 4.9 fails when* $d_1 + d_2$ *is even which is exactly when the vector bundle* $\mathcal{O}(d_1) \oplus \mathcal{O}(d_2) \to \mathbb{P}^2_k$ *is not relatively orientable. So this non-orientability is reflected in the combinatorics of the corresponding tropical curves.*

Finally, we want to review a very powerful application of tropical geometry to enumerative geometry. A major breakthrough of the use of tropical geometry in enumerative geometry was Mikhalkin's tropical correspondence theorem. Let's first describe the enumerative geometry problem. This concerns counting rational plane degree d curves through $n = 3d - 1$ points in general position. Call this number N_d.

Example 4.14. There is a unique degree 1 curve, that is a line, going through two points. So $N_1 = 1$. There is a unique conic, that is degree 2 plane rational curve, through 5 given point. So $N_2 = 1$. But after that N_d grows fast. For example $N_3 = 12$, $N_4 = 620$, $N_5 = 87304$, $N_6 = 26312976$, $N_7 = 14616808192$.

Definition 4.15. A plane tropical curve Γ is *nodal* if it has only 3 and 4-valent vertices. The genus of a plane tropical curve Γ is given by

$$g(\Gamma) = b_1(\Gamma) - \#\{\text{4-valent vertices}\}$$

where $b_1(\Gamma)$ denotes the first Betti number of Γ. A plane tropical curve Γ is *rational* if its genus is 0.

Milhalkin assigns a multiplicity $\text{mult}_{\mathbb{C}}(\Gamma)$ to a tropical curve Γ. Mikhalkin's tropical correspondence theorem provides to find N_d by translating the problem into a problem in tropical geometry with the following theorem.

Theorem 4.16 (Mikhalkin)**.**

$$N_d = N_d^{\text{trop}} := \sum_{\Gamma} \text{mult}_{\mathbb{C}}(\Gamma)$$

where the sum goes over all rational nodal tropical curves with Newton polygon Δ_d *through a configuration of* $3d - 1$ *points in general position.*

Example 4.17. There is a unique tropical line going through two given points in $\mathbb{R}^2$ if these points are in "tropical general position" (see Figure 12).

Remark 4.18. *If a point configuration of* $3d-1$ *points in* $\mathbb{R}^2$ *is "tropically in general position", then all curves through it are nodal.*

Next consider the problem over $k = \mathbb{R}$. In this case, we are interested in the count of real rational degree d plane curves through a configuration of n_1 real points and n_2 pairs of complex conjugate points with $n_1 + 2n_2 = n = 3d - 1$. Welschinger found that one should count such curve with a sign depending on the type of real nodes of the curve. For this observe that a real curve can have two types of nodes:

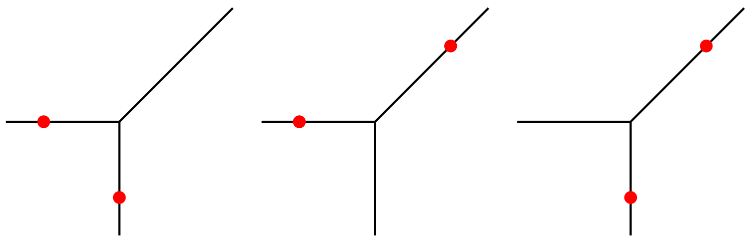
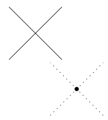


FIGURE 12. Tropical lines determined by two points in $\mathbb{R}^2$.

(1) a *split node* locally defined by $x^2 - y^2 = 0$

(2) a *solitary node* locally defined by $x^2 + y^2 = 0$



Definition 4.19. Define the *type* of a real node z to be

$$\text{type}(z) := \begin{cases} +1 & \text{if it is split} \\ -1 & \text{if it is solitary} \end{cases}$$

Then the *Welschinger sign* of a real cuve C is given by

$$\text{Wel}_{\mathbb{R}}(C) := \prod_{\text{real nodes } z} \text{type}(z).$$

Welschinger showed the following theorem [Wel05].

Theorem 4.20. *Let $\mathcal{P}$ be a configuration of n_1 real points and n_2 pairs of complex conjugate points in general position with $n_1 + 2n_2 = n = 3d - 1$. Then*

$$W_{d,n_2} := \sum_C \text{Wel}_{\mathbb{R}}(C)$$

where the sum goes over all real rational degree d curves through $\mathcal{P}$ is independent of the choice of point configuration.

Example 4.21. For example $W_{1,n_2} = 1$ and $W_{2,n_2} = 1$. When $d = 3$ we have $W_{3,n_2} = 8 - 2n_2$ for $n_2 = 0, 1, 2, 3, 4$.

Mikhalkin also provides a tropical correspondence theorem for the computation of $W_{d,0}$ (that is for point configurations consisting of only real points). Namely, he assigns another multiplicity $\text{mult}_{\mathbb{R}}(\Gamma)$ to a nodal tropical curve Γ and shows the following theorem.

Theorem 4.22 (Mikhalkin)**.**

$$W_{d,0} = W_{d,0}^{\text{trop}} := \sum_{\Gamma} \text{mult}_{\mathbb{R}}(\Gamma)$$

where the sum goes over all rational nodal tropical curves with Newton polygon Δ_d through a configuration of $3d - 1$ points in general position.

Remark 4.23. *The sums in Theorem 4.16 and Theorem 4.22 range over the same tropical curves, only the multiplicities differ.*

For W_{d,n_2} with $n_2 \geq 0$ Shustin defines a more general multiplicity $\text{mult}_{\mathbb{R}}(\Gamma)$ and provides a tropical correspondence theorem. However, the tropical curves considered here are different.

Theorem 4.24 (Shustin)**.**

$$W_{d,n_2} = W_{d,n_2}^{\text{trop}} = \sum \text{mult}_{\mathbb{R}}(\Gamma)$$

where the sum goes over all rational tropical curves with Newton polygon Δ_d through a configuration of n_1 "thin" (corresponding to real points) and n_2 "fat" points (corresponding to complex points) in $\mathbb{R}^2$ in tropical general position.

Remark 4.23 already indicates that there should be a count in $\mathrm{GW}(k)$ that specializes to both N_d and $W_{d,0}$.

In fact, this works according to recent work by Kass-Levine-Solomon-Wickelgren [KLSW23a, KLSW23b]. Suppose k is a perfect field of characteristic not equal to 2 or 3. Let $\overline{\mathcal{M}}_{0,n}(\mathbb{P}^2_k, d)$ be the moduli space of n-marked genus 0 stable maps to $\mathbb{P}^2$ of degree d. Let $\sigma = (L_1, \dots, L_r)$ be a list of finite field extensions L_i/k such that $\sum_{i=1}^r [L_i : k] = n$. There is a twisted evaluation map

$$\mathrm{ev}_\sigma \colon (\overline{\mathcal{M}}_{0,n}(\mathbb{P}^2_k, d))_\sigma \to \prod_{i=1}^r \mathrm{Res}_{L_i/k}\,\mathbb{P}^2_k.$$

Now take a point configuration of r points $p_1, \dots, p_r$ in general position with residue field $\kappa(p_i) = L_i$ for $i = 1, \dots, r$, that is a point in $\prod_{i=1}^r \mathrm{Res}_{L_i/k}\,\mathbb{P}^2_k$. Then in the preimage of this point configuration are exactly the stable maps with image a degree d rational plane curve through this point configuration, which is exactly what we want to count.

Theorem 4.25 (Kass-Levine-Solomon-Wickelgren)**.**

$$N^{\mathbb{A}^1}_{d,\sigma} := \deg^{\mathbb{A}^1}\left(\mathrm{ev}_\sigma \colon \overline{\mathcal{M}}_{0,n}(\mathbb{P}^2_k, d) \to \prod_{i=1}^r \mathrm{Res}_{L_i/k}\,\mathbb{P}^2_k\right)$$

is a well-defined element of $\mathrm{GW}(k)$. *Also,*

$$N^{\mathbb{A}^1}_{d,\sigma} = \deg^{\mathbb{A}^1}(\mathrm{ev}_\sigma) = \sum \deg^{\mathbb{A}^1}_u \mathrm{ev}_\sigma$$

can be written as the sum of local $\mathbb{A}^1$*-degrees over all stable maps that map to a curve passing through a given point configuration of* r *points* $p_1, \dots, p_r$ *in general position with residue field* $\kappa(p_i) = L_i$ *for* $i = 1, \dots, r$.

Recall that the Welschinger sign $\mathrm{Wel}_{\mathbb{R}}(C)$ depended on the type of real nodes. We want to generalize this.

Definition 4.26. Let C be a plane nodal curve over k and let z be a node of C. Then locally around z the curve C is given by an equation of the form $x^2 - \alpha y^2 = 0$ when base changing to the residue field $\kappa(z)$ of z. The *type* of z is $\mathrm{type}(z) := \alpha \in \kappa(z)^\times/(\kappa(z)^\times)^2$. The *quadratic weight* of C is

$$\mathrm{Wel}^{\mathbb{A}^1}_k(C) := \prod_{\text{nodes } z} N_{\kappa(z)/k}\langle \mathrm{type}(z)\rangle \in \mathrm{GW}(k)$$

where $N_{\kappa(z)/k}$ is the field norm.

Exercise 4.27. *Show that for a real curve* C *one has* $\mathrm{Wel}^{\mathbb{A}^1}_{\mathbb{R}}(C)) = \langle \mathrm{Wel}_{\mathbb{R}}(C)\rangle$ *in* $\mathrm{GW}(\mathbb{R})$.

The next theorem identifies the local $\mathbb{A}^1$-degrees in Theorem 4.25 with the quadratic weight of the image of the stable map.

Theorem 4.28 (Kass-Levine-Solomon-Wickelgren)**.**

$$\deg^{\mathbb{A}^1}_u \mathrm{ev}_\sigma = \mathrm{Tr}_{\kappa(u)/k}\left(\mathrm{Wel}^{\mathbb{A}^1}_{\kappa(u)}(u(C))\right)$$

where $\kappa(u)$ *is the residue field of the stable map* $u \colon C \to \mathbb{P}^2$ *in* $\overline{\mathcal{M}}_{0,n}(\mathbb{P}^2, d)$.

Consequently,

$$N^{\mathbb{A}^1}_{d,\sigma} = \sum \mathrm{Tr}_{\kappa(u)/k}\left(\mathrm{Wel}^{\mathbb{A}^1}_{\kappa(u)}(u(C))\right)$$

where the sum goes over all rational degree d plane curves through a configuration of r points in general position with residue field $L_1, \dots, L_r$.

Example 4.29. For $k=\mathbb{R}$ and $\sigma=(\mathbb{C},\dots,\mathbb{C},\mathbb{R},\dots,\mathbb{R})$ consisting of n_2 times $\mathbb{C}$ and n_1 times $\mathbb{R}$ with $n_1+2n_2=n$ we have that

$$\begin{aligned} N^{\mathbb{A}^1}_{\sigma,d} &= \sum_{\text{real curves C}} \mathrm{Wel}^{\mathbb{A}^1}_{\mathbb{R}}(C) + \sum_{\text{complex curves } C} \mathrm{Tr}_{\mathbb{C}/\mathbb{R}}(\mathrm{Wel}^{\mathbb{A}^1}(C)) \\ &= \sum_{\text{real curves C}} \mathrm{Wel}^{\mathbb{A}^1}_{\mathbb{R}}(C) + \sum_{\text{complex curves } C} \mathrm{Tr}_{\mathbb{C}/\mathbb{R}}(\langle 1\rangle) \\ &= \sum_{\text{real curves C}} \mathrm{Wel}^{\mathbb{A}^1}_{\mathbb{R}}(C) + \sum_{\text{complex curves } C} h \end{aligned}$$

which has signature

$$\begin{aligned} \mathrm{sgn}(N^{\mathbb{A}^1}_{\sigma,d}) &= \mathrm{sgn}\left(\sum_{\text{real curves C}} (\mathrm{Wel}^{\mathbb{A}^1}_{\mathbb{R}}(C)) + \sum_{\text{complex curves } C} h\right) \\ &= \mathrm{sgn}\left(\sum_{\text{real curves C}} \mathrm{Wel}_{\mathbb{R}}(C)\right) = W_{d,n_2} \end{aligned}$$

Now let's return to the computation of $N^{\mathbb{A}^1}_{d,\sigma}$. Let's first consider $\sigma=(k,\dots,k)$ with k a perfect field of characteristic not equal to 2 or 3. Then one can define

$$\mathrm{mult}^{\mathbb{A}^1}_k(\Gamma) := \begin{cases} \frac{\mathrm{mult}_{\mathbb{C}}(\Gamma)-1}{2}h + \langle \mathrm{mult}_{\mathbb{R}}(\Gamma)\rangle & \text{if } \mathrm{mult}_{\mathbb{R}}(\Gamma)\neq 0 \\ \frac{\mathrm{mult}_{\mathbb{C}}(\Gamma)}{2}h & \text{if } \mathrm{mult}_{\mathbb{R}}(\Gamma)=0 \end{cases}$$

which makes sense since $\mathrm{mult}_{\mathbb{R}}(\Gamma)\in\{-1,0,1\}$ and $\mathrm{mult}_{\mathbb{R}}(\Gamma)=0$ if and only if $\mathrm{mult}_{\mathbb{C}}(\Gamma)$ even. There is a tropical correspondence theorem for $\sigma=(k,\dots,k)$.

Theorem 4.30 (Jaramillo Puentes-Pauli)**.** *Let k be a perfect field of characteristic 0 or big (that is larger than the degrees of the polynomials). For $\sigma=(k,\dots,k)$ one has*

$$N^{\mathbb{A}^1}_{d,\sigma} = N^{\mathrm{trop}}_{d,\sigma} := \sum_{\Gamma} \mathrm{mult}^{\mathbb{A}^1}_k(\Gamma)$$

where the sum goes over all rational nodal tropical curves with Newton polygon Δ_d through a configuration of $3d-1$ points in general position.

A direct corollary is the following.

Corollary 4.31. *Let k be a perfect field of characteristic 0 or big. For $\sigma=(k,\dots,k)$ one has*

$$N^{\mathbb{A}^1}_{d,\sigma} = \frac{N_d - W_{d,0}}{2}h + W_{d,0}\langle 1\rangle$$

in $\mathrm{GW}(k)$.

Remark 4.32. *The sum in Theorem 4.30 is over the same tropical curves as in Theorem 4.16 and Theorem 4.22.*

What about other σ? Are there also tropical correspondence theorems similar to Shustin's theorem 4.24? There are first results in this direction in work in progress of Jaramillo Puentes-Markwig-Pauli-Röhrle [JPMPR24] for when $\sigma=(k(\sqrt{d_1}),\dots,k(\sqrt{d_{n_2}}),k,\dots,k)$ consists of quadratic and trivial field extensions.

Theorem 4.33 (Jaramillo Puentes-Markwig-Pauli-Röhrle)**.** *Let k be a perfect field of characteristic 0 or big and let $\sigma=(k(\sqrt{d_1}),\dots,k(\sqrt{d_{n_2}}),k,\dots,k)$. Then*

$$N^{\mathbb{A}^1}_{\sigma,d} = N^{\mathbb{A}^1,\mathrm{trop}}_{\sigma,d} := \sum_{\Gamma} \mathrm{mult}^{\mathbb{A}^1}_k(\Gamma)$$

where the sum goes over all rational tropical curves through a configuration of n_1 "thin" (corresponding to k-points) and n_2 "fat" points (corresponding to $k(\sqrt{d_i})$-points) in $\mathbb{R}^2$ in tropical general position.

Remark 4.34. *For* $k = \mathbb{R}$ *and* $\sigma = (\mathbb{C}, \ldots, \mathbb{C}, \mathbb{R}, \ldots, \mathbb{R})$*, taking signatures in* $N^{\mathbb{A}^1,\text{trop}}_{d,\sigma} = \text{mult}^{\mathbb{A}^1}_k(\Gamma)$ *recovers Shustin's tropical correspondence theorem 4.24.*

The techniques in [JPMPR24] in principle also work for general σ, but still have to be worked out.

Ackknowledgements

I would like to thank Nathan Tiggemann for catching many typos and suggesting improvements.

References

[BBE+23] Nikita Borisov, Thomas Brazelton, Frenly Espino, Thomas Hagedorn, Zhaobo Han, Jordy Lopez Garcia, Joel Louwsma, Wern Juin Gabriel Ong, and Andrew R. Tawfeek. $\mathbb{A}^1$-brouwer degrees in macaulay2, 2023.

[BBM+20] Thomas Brazelton, Robert Burklund, Stephen McKean, Michael Montoro, and Morgan Opie. The trace of the local $\mathbb{A}^1$-degree. *Homology Homotopy Appl.*, 23(1):243–255, 2020.

[BMP23] Thomas Brazelton, Stephen McKean, and Sabrina Pauli. Bézoutians and the $\mathbb{A}^1$-degree. *Algebra Number Theory*, 17(11):1985–2012, 2023.

[BS14] Erwan Brugallé and Kristin Shaw. A bit of tropical geometry. *Am. Math. Mon.*, 121(7):563–589, 2014.

[BW23] Tom Bachmann and Kirsten Wickelgren. Euler classes: six-functors formalism, dualities, integrality and linear subspaces of complete intersections. *J. Inst. Math. Jussieu*, 22(2):681–746, 2023.

[EH16] David Eisenbud and Joe Harris. *3264 and all that. A second course in algebraic geometry.* Cambridge: Cambridge University Press, 2016.

[EL77] David Eisenbud and Harold I. Levine. An algebraic formula for the degree of a C^∞ map germ. Sur une inegalite a la Minkowski pour les multiplicites. *Ann. Math. (2)*, 106:19–44, 1977.

[EMP24] Felipe Espreafico, Stephen McKean, and Sabrina Pauli. Quadratic segre involutions. Work in progress, 2024.

[FK13] Sergey Finashin and Viatcheslav Kharlamov. Abundance of real lines on real projective hypersurfaces. *Int. Math. Res. Not.*, 2013(16):3639–3646, 2013.

[Hat02] Allen Hatcher. *Algebraic topology.* Cambridge: Cambridge University Press, 2002.

[JPMPR24] Andrés Jaramillo Puentes, Hannah Markwig, Sabrina Pauli, and Felix Röhrle. Enriched tropical counts for quadratic extensions. Work in progress, 2024.

[JPP22] Andrés Jaramillo Puentes and Sabrina Pauli. Quadratically enriched tropical intersections, 2022.

[Khi77] G. N. Khimshiashvili. On the local degree of a smooth mapping. *Soobshch. Akad. Nauk Gruz. SSR*, 85:309–312, 1977.

[KLSW23a] Jesse Leo Kass, Marc Levine, Jake P. Solomon, and Kirsten Wickelgren. A quadratically enriched count of rational curves, 2023.

[KLSW23b] Jesse Leo Kass, Marc Levine, Jake P. Solomon, and Kirsten Wickelgren. A relative orientation for the moduli space of stable maps to a del pezzo surface, 2023.

[KW19] Jesse Leo Kass and Kirsten Wickelgren. The class of Eisenbud-Khimshiashvili-Levine is the local $\mathbb{A}^1$-Brouwer degree. *Duke Math. J.*, 168(3):429–469, 2019.

[KW21] Jesse Leo Kass and Kirsten Wickelgren. An arithmetic count of the lines on a smooth cubic surface. *Compos. Math.*, 157(4):677–709, 2021.

[Lam05] T. Y. Lam. *Introduction to quadratic forms over fields*, volume 67 of *Grad. Stud. Math.* Providence, RI: American Mathematical Society (AMS), 2005.

[McK21] Stephen McKean. An arithmetic enrichment of Bézout's theorem. *Math. Ann.*, 379(1-2):633–660, 2021.

[McK22] Stephen McKean. Circles of apollonius two ways, 2022.

[Mik05] Grigory Mikhalkin. Enumerative tropical algebraic geometry in $\mathbb{R}^2$. *J. Am. Math. Soc.*, 18(2):313–377, 2005.

[Mil65] John W. Milnor. *Topology from the differentiable viewpoint.* University Press of Virginia, Charlottesville, VA, 1965. Based on notes by David W. Weaver.

[MR] Grigory Mikhalkin and Johannes Rau. Tropical geometry. Lecture Notes.

[MS05] John W. Milnor and James D. Stasheff. *Characteristic classes*, volume 32 of *Texts Read. Math.* New Delhi: Hindustan Book Agency, 2005.

[MS15] Diane Maclagan and Bernd Sturmfels. *Introduction to tropical geometry*, volume 161 of *Grad. Stud. Math.* Providence, RI: American Mathematical Society (AMS), 2015.

[OT14] Christian Okonek and Andrei Teleman. Intrinsic signs and lower bounds in real algebraic geometry. *J. Reine Angew. Math.*, 688:219–241, 2014.

[Pau22] Sabrina Pauli. Quadratic types and the dynamic Euler number of lines on a quintic threefold. *Adv. Math.*, 405:37, 2022. Id/No 108508.

[PP24] Andrés Jaramillo Puentes and Sabrina Pauli. A quadratically enriched correspondence theorem, 2024.

[Sch63] Ludwig Schläfli. On the distribution of surfaces of the third order into species, in reference to the absence or presence of singular points, and the reality of their lines. *Philosophical Transactions of the Royal Society of London*, 153:193–241, 1863.

[Seg42] Beniamino Segre. *The Non-Singular Cubic Surfaces*. Oxford University Press, 1942.

[Shu06] Eugenii Shustin. A tropical calculation of the Welschinger invariants of real toric Del Pezzo surfaces. *J. Algebr. Geom.*, 15(2):285–322, 2006.

[SS75] Günter Scheja and Uwe Storch. Über Spurfunktionen bei vollständigen Durchschnitten. *J. Reine Angew. Math.*, 278/279:174–190, 1975.

[Wel05] Jean-Yves Welschinger. Invariants of real symplectic 4-manifolds and lower bounds in real enumerative geometry. *Invent. Math.*, 162(1):195–234, 2005.

[WW20] Kirsten Wickelgren and Ben Williams. Unstable motivic homotopy theory. In *Handbook of homotopy theory*, pages 931–972. Boca Raton, FL: CRC Press, 2020.