Enumerative Geometry and $\mathbb{A}^1$-homotopy theory

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Bézout's theorem-a first example

One basic principle in enumerative geometry:

A sum of local invariants = a global invariant

Theorem (Bézout)

Let C_1, C_2 be curves in $\mathbb{P}^2$ of degree d_1, d_2, without common component. Then

$$\sum_{p \in C_1 \cap C_2} m_p(C_1 \cdot C_2; \mathbb{P}^2) = d_1 d_2$$

Bézout's theorem

To explain:

C_1, C_2 are the solutions of homogeneous polynomials $F_1(x, y, z), F_2(x, y, z)$ of degree d_1, d_2 in

$$\mathbb{P}^2 = \{(x, y, z) \neq (0, 0, 0)\}/(x, y, z) \sim (tx, ty, tz), t \neq 0$$

At $p \in C_1 \cap C_2$, $m_p(C_1 \cdot C_2; \mathbb{P}^2)$ is the *intersection multiplicity* at p:

s, t local coordinates at p, $f_i \in k(p)[[s, t]]$ local defining equation for C_i, then

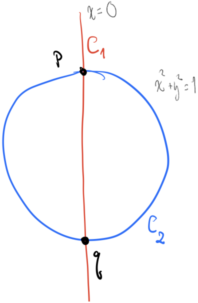
$$m_p(C_1 \cdot C_2; \mathbb{P}^2) := \dim_k k(p)[[s, t]]/(f_1, f_2) < \infty.$$

Bézout's theorem

Case 1: $F_1 = x$, $F_2 = x^2 + y^2 - z^2$, $p = (0, 1, 1)$, $q = (0, -1, 1)$. Let $f_i(x, y) = F_i(x, y, 1)$. Then

$$\begin{aligned} m_p(C_1 \cdot C_2) &:= \dim_k k[[x, y-1]]/(f_1, f_2) \\ &= \dim_k k[[x, y-1]]/(x, x^2 + (y-1)(y+1)) \\ &=^* \dim_k k[[y-1]]/(y-1) = \dim_k k = 1, \end{aligned}$$

The equality $=^*$ follows because $y + 1$ is a unit in the power series ring $k[[x, y-1]]$.

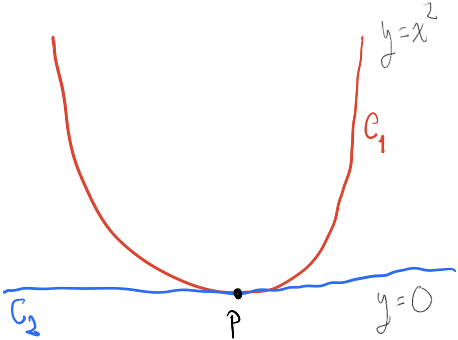


Similarly, $m_q(C_1 \cdot C_2) = 1$: *transverse intersections.*

Bézout's theorem

Case 2: $F_1 = yz - x^2$, $F_2 = y$, $p = (0, 0, 1)$. Then

$$m_p(C_1 \cdot C_2) := \dim_k k[[x,y]]/(f_1, f_2) = \dim_k k[[x,y]]/(y - x^2, y)$$
$$= \dim_k k[[x]]/(x^2) = \dim_k k \oplus k \cdot x = 2.$$

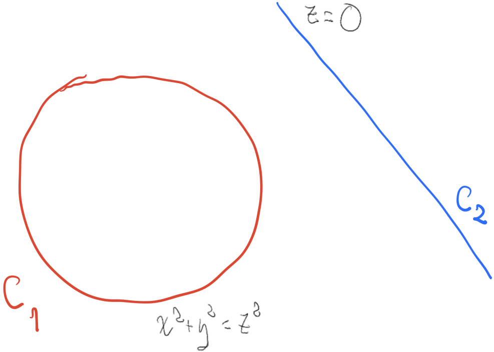


Bézout's theorem

Case 3: $k = \mathbb{R}$, $F_1 = x^2 + y^2 - z^2$, $F_2 = z$. $z = 0 \Rightarrow y \neq 0$.

Take $f_i = F_i(x, 1, z)$, so $\mathbb{R}[x, z]/(f_1, f_2) = \mathbb{R}[x]/(x^2 + 1) = \mathbb{C} \Rightarrow$ a single point of intersection p with $\mathbb{R}(p) = \mathbb{C}$.

$$m_p(C_1 \cdot C_2) := \mathbb{R}[x, z]/(f_1, f_2) = \dim_{\mathbb{R}} \mathbb{C} = 2.$$



Bézout's theorem

$$\text{Case } 1 : C_1 \cap C_2 = \{p, q\},\ d_1 = 2, d_2 = 1,\ m_p + m_q = 2 = d_1 d_2,$$
$$\text{Case } 2 : C_1 \cap C_2 = \{p\},\ d_1 = 2, d_2 = 1,\ m_p = 2 = d_1 d_2,$$
$$\text{Case } 3 : C_1 \cap C_2 = \{p\},\ d_1 = 2, d_2 = 1,\ m_p = 2 = d_1 d_2,$$

verifying Bézout's theorem

Theorem (Bézout)

Let C_1, C_2 be curves in $\mathbb{P}^2$ of degree d_1, d_2, without common component. Then

$$\sum_{p \in C_1 \cap C_2} m_p(C_1 \cdot C_2; \mathbb{P}^2) = d_1 d_2$$

in these cases.

Bézout's theorem

There is an extension to intersections of hypersurfaces in $\mathbb{P}^n$.

Theorem (Bézout)

Let $H_1, \ldots, H_n$ be hypersurfaces in $\mathbb{P}^n$ of degree $d_1, \ldots, d_n$. Suppose the intersection $H_1 \cap \ldots \cap H_n$ is a finite set Then

$$\sum_{p \in \cap_{i=1}^n H_i} m_p(H_1 \cdots H_n; \mathbb{P}^n) = d_1 \cdots d_n$$

As for $\mathbb{P}^2$, there is an algebraic formula for the intersection multiplicity $m_p(H_1 \cdots H_n; \mathbb{P}^n)$.

The Chow ring

One can systematize these intersection formulas, and also give an easy proof of the Bézout theorems, by introducing a purely algebraic analog of the cohomology ring: The *Chow ring* of algebraic cycles modulo rational equivalence.

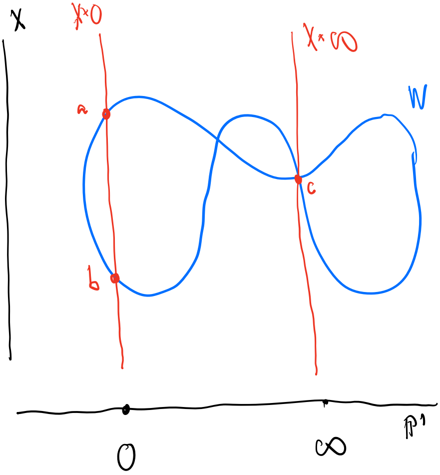
Definition

Let X be a smooth algebraic variety over a field k (of pure dimension n).

1. The group $Z^i(X)$ of codimesion i algebraic cycles on X is the free abelian group on the (reduced) subvarieties W of X of codimension i.
2. The subgroup $R^i(X) \subset Z^i(X)$ is generated by cycles of the form $W \cdot (X \times 0 - X \times \infty)$ for $W \subset X \times \mathbb{P}^1$ a codimension i subvariety, not contained in $X \times \{0, \infty\}$.
3. The Chow group of codimension i cycles modulo rational equivalence is

$$\mathrm{CH}^i(X) := Z^i(X)/R^iX).$$

The Chow ring-a picture



$1\cdot a + 1\cdot b \sim 2\cdot c$ in $CH^*(X)$

The Chow package

The Chow groups fit together to form a graded group $\mathrm{CH}^*(X) := \oplus_{i=0}^{\dim_k X}\mathrm{CH}^i(X)$ with the following properties:

1. There is an *intersection product*

$$\mathrm{CH}^i(X) \times \mathrm{CH}^j(X) \to \mathrm{CH}^{i+j}(X)$$

making $\mathrm{CH}^*(X)$ a commutative, graded ring with unit $1_X = [X]$.

2. For $f : Y \to X$ a map of smooth k-varieties, there is a *pull-back* map

$$f^* : \mathrm{CH}^*(X) \to \mathrm{CH}^*(Y)$$

which is a ring homomorphism.

3. For $f : Y \to X$ a proper map of smooth k-varieties, there is a *push-forward map*

$$f_* : \mathrm{CH}^i(Y) \to \mathrm{CH}^{i+\dim_k X-\dim_k Y}(X).$$

The Chow package

4. For $V \to X$ a rank r vector bundle on X, we have *Chern classes*

$$c_i(V) \in \mathrm{CH}^i(X),\ i = 1, \ldots, r.$$

The top Chern class is given by

$$c_r(V) = s_2^* s_{1*}(1_X)$$

where $s_1, s_2 : X \to V$ are any two sections ($s_1 = s_2$ is also allowed), and

$$c_1(L \otimes M) = c_1(L) + c_1(M).$$

5. For X irreducible, $\mathrm{CH}^0(X) = \mathbb{Z}$, generated by $1_X := [X]$.

6. For $p_X : X \to \mathrm{Spec}\, k$ proper of dimenison n over k, the push-foward map $p_{X*} : \mathrm{CH}^n(X) \to \mathrm{CH}^0(\mathrm{Spec}\, k)$ is called the *degree map*. Explicity, for $\sum_{p \in X} n_p[p] \in \mathrm{CH}^n(X)$, we have

$$\deg_k(\sum_{p \in X} n_p[p]) = \sum_p n_p \cdot [k(p) : k].$$

The Chow package

The ring structure is (more or less) defined as we did above, for each irreducible component Z of an intersection $W_1 \cap W_2$, one has an algebraic formula for the multiplicity $m_Z(W_1 \cdot W_2; X) \in \mathbb{Z}$ and one sets

$$W_1 \cdot W_2 := \sum_{Z \text{ an irred. comp. of } W_1 \cap W_2} m_Z(W_1 \cdot W_2; X) \cdot Z$$

This only makes sense if the intersection is proper:

$$\mathrm{codim}_X Z = \mathrm{codim}_X W_1 + \mathrm{codim}_X W_2$$

and reducing to this case entails some technical difficulties.

Proof of Bézout's theorem

1. For $d \in \mathbb{N}$, there is a line bundle $O_{\mathbb{P}^n}(d) \to \mathbb{P}^n$ with global sections the degree d homogeneous polynomials in $x_0, \ldots, x_n$. One has $O_{\mathbb{P}^n}(d) = O_{\mathbb{P}^n}(1)^{\otimes d}$.

2. Given $H_1, \ldots, H_n \subset \mathbb{P}^n$, hypersurfaces of degree $d_1, \ldots, d_n$, the defining equations F_i of H_i define a section

$$s := (F_1, \ldots, F_n) : \mathbb{P}^n \to \oplus_{i=1}^n O_{\mathbb{P}^n}(d_i).$$

Proof of Bézout's theorem

3. Let $s_0 : \mathbb{P}^n \to \oplus_{i=1}^n O_{\mathbb{P}^n}(d_i)$ be the 0-section. From general principles, we have

$$c_n(\oplus_{i=1}^n O_{\mathbb{P}^n}(d_i)) = s^* s_{0*}([\mathbb{P}^n]) = H_1 \cdots H_n \in \mathrm{CH}^n(\mathbb{P}^n),$$

$$c_n(\oplus_{i=1}^n O_{\mathbb{P}^n}(d_i)) = \prod_{i=1}^n c_1(O_{\mathbb{P}^n}(d_i)) \in \mathrm{CH}^n(\mathbb{P}^n),$$

and

$$c_1(O_{\mathbb{P}^n}(d_i)) = d_i \cdot c_1(O_{\mathbb{P}^n}(1)),$$

so

$$c_n(\oplus_{i=1}^n O_{\mathbb{P}^n}(d_i)) = d_1 \cdots d_n \cdot c_1(O_{\mathbb{P}^n}(1))^n$$

4. Now repeat with $d_1 = \ldots = d_n = 1$, and take $F_i = x_i$, we get

$$\deg_k c_1(O_{\mathbb{P}^n}(1))^n = \deg_k(x_1 = \ldots = x_n = 0) = \deg_k((1, 0, \ldots, 0)) = 1$$

Putting this all together and applying $deg_k(-)$ gives

$$\deg_k(H_1 \cdots H_n) = d_1 \cdots d_n.$$

Other applications

1. One can count the (expected) number of lines on a smooth degree d hypersurface in $\mathbb{P}^n$, with $d = 2n - 3$, as

$$\#\{\ell \subset X\} = \deg c_{d+1}(\mathrm{Sym}^d E_2^\vee)$$

where $E_2 \to \mathrm{Gr}(2, n+1)$ is the tautological bundle. One can work out an explicit formula for this using the Schubert calculus (intersection theory on the Grassmannian).

$$(d, n) = (3, 3) \Rightarrow 27 \text{ lines on a cubic surface in } \mathbb{P}^3.$$

$$(d, n) = (5, 4) \Rightarrow 2875 \text{ lines on a quintic threefold in } \mathbb{P}^4.$$

Other applications

2. One has an algebraic version of the Gauß-Bonnet theorem

Theorem (algebraic Gauß-Bonnet)

Let X be a smooth proper variety of dimension d over a field k, with tangent bundle $T_X \to X$. Then we have $c_d(T_X) \in \mathrm{CH}^d(X)$ and

$$\deg_k c_d(T_X) = \chi^{top}(X)$$

Other applications

3. Counting rational curves of degree d through $3d-1$ points in $\mathbb{P}^2$. This number, N_d, is the degree of the "evaluation map"

$$ev : \bar{\mathcal{M}}_{0,n}(\mathbb{P}^2, d) \to (\mathbb{P}^2)^n, \quad n = 3d-1,$$

i.e.

$$ev_*([\bar{\mathcal{M}}_{0,}(\mathbb{P}^2, d)]) = N_d \cdot [(\mathbb{P}^2)^n].$$

Using intersection theory, Kontsevich derived a recursive formula

$$N_d = \sum_{d_1, d_2 > 0, d_1 + d_2 = d} N_{d_1} N_{d_2} \left(d_1^2 d_2^2 \binom{3d-4}{3d_1-2} - d_1^3 d_2 \binom{3d-4}{3d_1-1} \right).$$

Summary: the Chow package

To summarize: one has a $\mathbb{Z}$-valued enumerative geometry on k-varieties based on:

1. The Chow ring with its intersection product and Chern classes of vector bundles

2. The pull-back maps for arbitrary morphisms and push-forward maps for proper maps, in particular, the degree map

$$\deg_k : \mathrm{CH}^n(X) \to \mathrm{CH}^0(\mathrm{Spec}\, k) = \mathbb{Z}$$

for X smooth and proper of dimension n over k.

Enter $\mathbb{A}^1$-homotopy theory

The basic principle: Using $\mathbb{A}^1$-homotopy theory as a foundation, one can construct a new "intersection theory package" where the degree map has values in the Grothendieck-Witt ring of non-degenerate quadratic forms over k.

There is a drawback: one does not have a full theory of Chern classes, but only an Euler class, which refines the top Chern class.

There is a complication: the resulting package resembles the structure of integral cohomology on not-necessarily oriented manifolds, so the groups replacing the Chow groups come with "twisted" versions with respect to invertible sheaves, playing the role of orientation bundles.

The twisting plays a role in the "wrong way" pushforward maps. To get well-defined invariants by applying a pushforward, one needs to construct compatibilities between various orientations.

The Grothendieck-Witt ring and the Witt ring

k a field (characteristic $\neq 2$). We have the notion of a non-degenerate quadratic form on a finite dimensional k-vector space V

$$q : V \to k$$

Roughly: choosing a basis for V, q is represented by a symmetric matrix B by

$$q(x) = x^t Bx$$

and q is non-degenerate if $\det B \neq 0$.
We have the relation of isometry, induced on matrices by

$$B \sim S^t BS,$$

S invertible.

The Grothendieck-Witt ring and the Witt ring

We can add quadratic forms via block sum of matrices

$$(B_1, B_2) \mapsto \begin{pmatrix} B_1 & 0 \\ 0 & B_2 \end{pmatrix}$$

and multiply by the "Kronecker product"

$$(B_1, B_2) \mapsto B_1 \otimes B_2$$

These operations make the set of isometry classes into a commutative semi-ring (there is no additive inverse) so we formally introduce one by taking formal differences $q_1 - q_2$ with $(q_1 + q) - (q_2 + q) = q_1 - q_2$.

This produces the Grothendieck-Witt ring $\mathsf{GW}(k)$.

Sending (q, V) to $\dim_k V$ defines the rank homomorphism

$$\mathrm{rank} : \mathsf{GW}(k) \to \mathbb{Z}$$

Let $I = \ker(\mathrm{rank})$.

The Grothendieck-Witt ring and the Witt ring

There is a particularly simple quadratic form, the hyperbolic form $H(x, y) = x^2 - y^2$. This has the property that $q \cdot H = \mathrm{rank}(q) \cdot H$, so the ideal (H) is the subgroup $\mathbb{Z} \cdot H$. The *Witt ring* $W(k)$ is

$$W(k) := \mathrm{GW}(k)/(H).$$

In fact:

$$W(k) = \{\text{non-degenerate, anisotropic quadratic forms over } k\}/\text{isometry}$$

The Grothendieck-Witt sheaf and the Witt sheaf

Replacing k with a commutative ring R gives $\mathrm{GW}(R)$ and $W(R)$, functorial for ring homomorphisms. This defines sheaves $\mathcal{GW}$ and $\mathcal{W}$ on the category $\mathbf{Sm}/k$ of smooth k-varieties.

A section of $\mathcal{GW}$ over some X is given locally as a formal difference of vector bundles $V \to X$ with a (fiberwise) non-degenerate quadratic form $q : V \to O_X$. Replacing the trivial line bundle O_X with an arbitrary line bundle L defines the sheaf $\mathcal{GW}_X(L)$ on X. Similarly for $\mathcal{W}_X(L)$.

The kernel $\mathcal{I}$ of the rank map $\mathcal{GW} \to \mathbb{Z}$ is a sheaf of $\mathcal{GW}$-ideals.

This is all quite classical. Now for the $\mathbb{A}^1$-homotopy theory.

Milnor Witt sheaves

For a field k, we have the *motivic stable homotopy category* $\mathrm{SH}(k)$, a symmetric monoidal category with unit object $\mathbb{S}_k$, the *motivic sphere spectrum*. We will treat $\mathrm{SH}(k)$ as the blackest of black boxes.

Given an object $\mathcal{E} \in \mathrm{SH}(k)$ and a pair of integers a, b, we have a sheaf $\pi_{a,b}(\mathcal{E})$ on $\mathbf{Sm}/k$. This is an analog of the stable homotopy groups $\pi_n^s(E)$ for a spectrum E.

Since we have two indices, a, b, the motivic analog of π_0^s is the graded sheaf $\oplus_n \pi_{n,n}$.

Definition (Theorem?)

The sheaf $\pi_{-n,-n}(\mathbb{S}_k)$ on $\mathbf{Sm}/k$ is the *Milnor-Witt K-sheaf* $\mathcal{K}_n^{MW}$.

In fact, for a field F, Mike Hopkins and Fabien Morel have defined by generators and relations the graded algebra $K_*^{MW}(F)$, and Morel shows how these extend to the sheaf $\mathcal{K}_*^{MW}$. So the definition is really a theorem of Morel's.

There is an important element $\eta \in \mathcal{K}_{-1}^{MW}$: the *algebraic Hopf map*.

Milnor Witt sheaves

For a field F, one also has the Milnor K-theory of F, defined by Milnor in 1970

$$K_*^M(F) := (F^\times)^{\otimes_{\mathbb{Z}} *}/\langle\{a \otimes (1-a) \mid a \in F \setminus \{0,1\}\rangle$$

which extends to a sheaf of $\mathbb{N}$-graded rings $\mathcal{K}_*^M$ on $\mathbf{Sm}/k$.

Theorem (Morel)

1. $\mathcal{K}_0^{MW} \cong \mathcal{GW}$
2. For all $n < 0$, $\mathcal{K}_n^{MW} \cong \mathcal{W}$.
3. There is a surjection of sheaves of graded rings $\mathcal{K}_*^{MW} \to \mathcal{K}_*^M$ with kernel $\eta \cdot \mathcal{K}_*^{MW}$. For $n \geq 0$ the kernel of $\mathcal{K}_n^{MW} \to \mathcal{K}_n^M$ is $\mathcal{I}^{n+1}$.
4. The map $\times\eta : \mathcal{K}_n^{MW} \to \mathcal{K}_{n-1}^{MW}$ induces the inclusion $\mathcal{I}^{n+1} \subset \mathcal{I}^n$ for $n \geq 1$, the surjection $\mathcal{GW} \to \mathcal{W}$ for $n = 0$ and the identity on $\mathcal{W}$ for $n < 0$.

Twisted Milnor Witt sheaves

Now let $L \to X$ be a line bundle on $X \in \mathbf{Sm}/k$. $\mathcal{GW}_X(L)$ is a module for $\mathcal{GW}_X$ and $\mathcal{K}_n^{MW}$ is a module for $\mathcal{K}_0^{MW} = \mathcal{GW}$.

Definition

The L-twised Milnor-Witt sheaf on X is

$$\mathcal{K}_n^{MW}(L) := \mathcal{K}_n^{MW} \otimes_{\mathcal{GW}} \mathcal{GW}(L).$$

Chow-Witt groups

Definition

For $X \in \mathbf{Sm}/k$ with line bundle L, and integer $n \geq 0$, define the nth L-twisted Chow-Witt group of X by

$$\widetilde{\mathrm{CH}}^n(X, L) := H^n(X, \mathcal{K}_n^{MW}(L))$$

This is a reasonable definition because of *Bloch's formula*

Theorem (Kato, Elbaz-Vincent and Müller-Stach, Kerz)

For $X \in \mathbf{Sm}/k$

$$\mathrm{CH}^n(X) \cong H^n(X, \mathcal{K}_n^M).$$

The map of sheaves $\mathcal{K}_n^{MW}(L) \to \mathcal{K}_n^M$ defines the comparison map

$$\widetilde{\mathrm{CH}}^n(X, L) \to \mathrm{CH}^n(X).$$

The Chow-Witt package

1. $\widetilde{\mathrm{CH}}^n(X, L)$ is functorial in X and L: given $\rho : L \xrightarrow{\sim} L'$, we have $\rho_* : \widetilde{\mathrm{CH}}^n(X, L) \xrightarrow{\sim} \widetilde{\mathrm{CH}}^n(X, L')$. Given $f : Y \to X$ in $\mathbf{Sm}/k$ we have $f^* : \widetilde{\mathrm{CH}}^n(X, L) \to \widetilde{\mathrm{CH}}^n(Y, f^*L)$. Also, there is a canonical isomorphism

$$\widetilde{\mathrm{CH}}^n(X, L) \cong \widetilde{\mathrm{CH}}^n(X, L \otimes M^{\otimes 2}).$$

for any line bundle $M \to X$.

2. There is a multiplication map $\widetilde{\mathrm{CH}}^n(X, L) \times \widetilde{\mathrm{CH}}^m(X, L') \to \widetilde{\mathrm{CH}}^{n+m}(X, L \otimes L')$ making $\oplus_{n, L \in \mathrm{Pic}(X)} \widetilde{\mathrm{CH}}^n(X, L)$ a $\mathbb{Z} \times \mathrm{Pic}(X)$ graded ring.

3. For $f : Y \to X$ proper in $\mathbf{Sm}/k$, let $\omega_f := \omega_{Y/k} \otimes f^* \omega_{X/k}^{-1}$. There is a push-forward map

$$f_* : \widetilde{\mathrm{CH}}^n(Y, f^*L \otimes \omega_f) \to \widetilde{\mathrm{CH}}^{n + \dim_k X - \dim_k Y}(X, L)$$

The Chow-Witt package

4. $\widetilde{\mathrm{CH}}^n(\mathrm{Spec}\, k) = 0$ for $n > 0$ and $\widetilde{\mathrm{CH}}^0(\mathrm{Spec}\, k) = \mathrm{GW}(k)$. For $p_X : X \to \mathrm{Spec}\, k$ in $\mathbf{Sm}/k$, proper of dimension d over k, define the quadratic degree map

$$\widetilde{\deg}_k : \widetilde{\mathrm{CH}}^d(X, \omega_{X/k}) \to \widetilde{\mathrm{CH}}^0(\mathrm{Spec}\, k) = \mathrm{GW}(k)$$

to be p_{X*}.

5. The structures (1)-(4) are compatible with the corresponding ones on CH^* via the comparison maps.

The Chow-Witt package

Definition

Let $V \to X$ be a rank r vector bundle on $p_X : X \to \operatorname{Spec} k$ in $\mathbf{Sm}/k$. Let $s_0 : X \to V$ be the zero-section and let $1_X = p_X^*(1) \in \widetilde{\mathrm{CH}}^0(X)$. . Define the Euler class of V by

$$e(V) := s_0^* s_{0*}(1_X) \in \widetilde{\mathrm{CH}}^r(X, \det(V)^{-1}).$$

The comparison map sends $e(V) \in CHW^r(X, \det(V)^{-1})$ to $c_r(V) \in \mathrm{CH}^r(X)$.

We also have $e(V) = s_2^* s_{1*}(1_X)$ for any two sections s_1, s_2 of V.

Warning! We do not have an analog of the Chern classes $c_i(V)$ for $i < r$.

Applications

With the Chow-Witt package, one can refine many results from the $\mathbb{Z}$-valued enumerative geometry to yield quadratic invariants from geometric problems.

The main thread of proofs remains the same, except that one needs to work on the twists whenever a pushforward map shows up.

Why bother?

The rank map $\mathrm{GW}(\mathbb{C}) \to \mathbb{Z}$ is an isomorphism, while $\mathrm{GW}(\mathbb{R})$ maps injectively to $\mathbb{Z} \times \mathbb{Z}$ by $(\mathrm{rank}, \text{signature})$. For enumerative problems, applying the rank map to the quadratic degree recover the classic integer count and applying the signature gives information on the real counts.

In general the GW-valued invariants carry some additional arithmetic information missing in the classical $\mathbb{Z}$-valued invariants.

Applications

Theorem (Quadratic Bézout-S. McKean)

Given C_1, C_2 curves in $\mathbb{P}^2$ of degrees d_1, d_2, without common component. Suppose $d_1 + d_2$ is odd. One defines quadratic intersection multiplicities

$$m_p(C_1 \cdot C_2) \in \mathrm{GW}(k(p))$$

and

$$\sum_{p \in C_1 \cap C_2} \mathrm{Tr}_{k(p)/k} m_p(C_1 \cdot C_2) = \frac{d_1 d_2}{2} \cdot H$$

The quadratic intersection multiplicity at $p \in C_1 \cap C_2$, in the case of a transverse intersection, is the quadratic form $x \mapsto ux^2$, where

$$u = \det(\partial f_i / \partial t_j)(p)$$

with f_i a local defining equation and x_1, x_2 are local coordinates, suitably chosen to respect an orientation condition.

Applications

Proof of Quadratic Bézout:

The idea is similar to the classical case. We have

$$e(\mathcal{O}(d_1) \oplus \mathcal{O}(d_2)) \in \widetilde{\mathrm{CH}}^2(\mathbb{P}^2, \mathcal{O}(-d_1 - d_2)).$$

Also $\omega_{\mathbb{P}^2} = \mathcal{O}(-3)$, so

$$\mathcal{O}(-d_1 - d_2) \cong \omega_{\mathbb{P}^2} \otimes \mathcal{O}(\frac{-d_1 - d_2 + 3}{2})^{\otimes 2}$$

We have the degree map

$$\widetilde{\mathrm{CH}}^2(\mathbb{P}^2, \mathcal{O}(-d_1 - d_2)) \cong \widetilde{\mathrm{CH}}^2(\mathbb{P}^2, \omega_{\mathbb{P}^2} \otimes \mathcal{O}(\frac{-d_1 - d_2 + 3}{2})^{\otimes 2}))$$
$$\cong \widetilde{\mathrm{CH}}^2(\mathbb{P}^2, \omega_{\mathbb{P}^2}) \xrightarrow{\widetilde{\deg}_k} \mathrm{GW}(k)$$

giving

$$\widetilde{\deg}_k(e(\mathcal{O}(d_1) \oplus \mathcal{O}(d_2))) = \widetilde{\deg}_k(e(\mathcal{O}(d_1)) \cdot e(\mathcal{O}(d_2))) \in \mathrm{GW}(k).$$

Applications

The multiplication $\eta : \mathcal{K}_n^{MW} \to \mathcal{K}_{n-1}^{MW}$ satisfies

$$\eta \cdot e(L) = 0,$$

for any line bundle L, so

$$\eta \cdot \widetilde{\deg}_k(e(O(d_1)) \cdot e(O(d_2))) = \widetilde{\deg}_k(\eta \cdot e(O(d_1)) \cdot e(O(d_2))) = 0.$$

But $\eta : \mathrm{GW}(k) \to W(k)$ is the quotient map with kernel $\mathbb{Z} \cdot H$, so

$$\widetilde{\deg}_k(e(O(d_1) \oplus O(d_2))) = m \cdot H$$

for some m. Then $m = (d_1 d_2)/2$ by taking the rank and comparing with the classical Bézout theorem.

Applications

Theorem (Quadratic Gauß-Bonnet. L., Déglise-Jin-Khan)

There is a quadratic Euler characteristic of a smooth proper variety X over a field k, $\chi(X/k) \in \mathrm{GW}(k)$. The rank recovers $\chi^{top}(X)$, and for $k = \mathbb{R}$, the signature recovers $\chi^{top}(X(\mathbb{R}))$.
Moreover

$$\chi(X/k) = \widetilde{\deg}_k(e(T_X)).$$

Applications

Example (Quadratic count of lines on a hypersurface)

For $d = 2n-3$, it turns out that $\det \mathrm{Sym}^d E_2 = \omega_{\mathrm{Gr}(2,n+1)/k} \otimes M^{\otimes 2}$ for some line bundle M on $\mathrm{Gr}(2, n+1)$. So we have a quadratic count of the lines on $X_{2n-3} \subset \mathbb{P}^n$, given by $\widetilde{\deg}_k(e(\mathrm{Sym}^d E_2^\vee))$, using a similar composition as for the quadratic Bézout theorem.

In fact, the image of $\widetilde{\deg}_k(e(\mathrm{Sym}^d E_2^\vee))$ in $W(k)$ is simply $d!! \in W(k)$, $d!! := d(d-2)\cdots 3\cdot 1$, so

$$\widetilde{\deg}_k(e(\mathrm{Sym}^d E_2^\vee)) = d!! + \frac{n_d - d!!}{2} H$$

where $n_d = \deg_k c_{d+1}(\mathrm{Sym}^d E_2^\vee)$ is the classical count.

The case $n = 3$ (lines on a cubic surface in $\mathbb{P}^3$) was first computed by Kass-Wickelgren. Combined with the classical count of 27 lines, this yields

$$\widetilde{\deg}_k(e(\mathrm{Sym}^d E_2^\vee)) = 3 + 12 \cdot H \in \mathrm{GW}(k).$$

Applications

Example (Quadratic counts of rational curves-Kass-L.-Solomon-Wickelgren)

We show that the evaluation map $ev : \bar{\mathcal{M}}_{0,n}(\mathbb{P}^2, d) \to (\mathbb{P}^2)^n$ has a canonical orientation:

$$\omega_{ev} \cong L^{\otimes 2}.$$

This gives an isomorphism

$$\begin{aligned} H^0(\bar{\mathcal{M}}_{0,n}(\mathbb{P}^2, d), \mathcal{GW}) &\cong H^0(\bar{\mathcal{M}}_{0,n}(\mathbb{P}^2, d), \mathcal{GW}(L^{\otimes 2})) \\ &\cong H^0(\bar{\mathcal{M}}_{0,n}(\mathbb{P}^2, d), \mathcal{GW}(\omega_{ev})), \end{aligned}$$

so we can define

$$ev_*(1_{\bar{\mathcal{M}}_{0,n}(\mathbb{P}^2,d)}) \in H^0((\mathbb{P}^2)^n, \mathcal{GW}) \cong \mathrm{GW}(k)$$

This recovers the classical count on taking the rank and the signature gives Welchinger's real curve count. This extends to (most) del Pezzo surfaces.

Applications

Example (Quadratic counts of rational curves)

Sabrina Pauli and Andrés Jaramillo Puentes have used tropical methods to give explicit computations for toric del Pezzo surfaces. Jaramillo Puentes has given "wall-crossing formulas" in the tropical setting, and Erwan Brugallé-Kirsten Wickelgren have another wall-crossing formula in a more geometric setting.

Example (Counting linear spaces)

Thomas Brazelton has given quadratic versions of the classical count of the number of p-planes in (m+p)-space intersecting mp p-planes in general position.

Applications

Example (Localization methods)

Quadratic analogs of torus localization and the Bott residue formula have been applied by L. and Pauli to give quadratic counts of twisted cubics on hypersurfaces and complete intersections.

Virtual localization formulas have been used by A. Viergever to compute some quadratic versions of DT invariants.

Example (Quadratic knot invariants)

The quadratic machinery has been applied by Clémentine Lemarié--Rieusset to define quadratic linking numbers.
Mario Kummer and Daniele Agostini have defined an "arithmetic writhe", an invariant of an embedded $\mathbb{P}^1$ in a $\mathbb{P}^3$ (with orientation data). This not only refines a classical construction for knots, but resolves a lack-of-independence problem with the classical construction.

And many others: Jesse Pajwani, Ambrus Pál, Tom Bachmann, Candace Bethea, Simon Pepin Lehalleur, Srivinvas, Ran Azouri, Lukas Bröring, ...