4. Lecture 4: Tropical enumerative geometry

Exercise 4.1. Find the enriched tropical curves defined by the polynomials in Exercise 3.1 and compute the $\mathbb{A}^1$-intersection multiplicities at their intersections.

Exercise 4.2. Show the following.

(1) There is a bijection

$$k\{\{t\}\}^\times/(k\{\{t\}\}^\times)^2 \cong k^\times/k^\times,\ a(t) = a_0 t^{q_0} + h.o.t. \mapsto a_0$$

(2) Conclude that $\mathrm{GW}(k\{\{t\}\}) \to \mathrm{GW}(k)$ defined by $\langle a_0 t^{q_0} + h.o.t.\rangle \mapsto \langle a_0\rangle$ is an isomorphism by checking that this map respects the relations in the Grothendieck-Witt rings.

Exercise 4.3. Let C_1 and C_2 be two curves in $\mathbb{P}^1 \times \mathbb{P}^1$ defined by f_1 and f_2 of bidegree (d_1, d_2) and (e_1, e_2), respectively. Then f_1 and f_2 define a section of

$$V := \mathcal{O}(d_1, d_2) \oplus \mathcal{O}(e_1, e_2) \longrightarrow \mathbb{P}^1 \times \mathbb{P}^1$$

where $\mathcal{O}(a, b) = \pi_1^*\mathcal{O}_{\mathbb{P}^1}(a) \otimes \pi_2^*\mathcal{O}_{\mathbb{P}^1}(b)$ and $\pi_i : \mathbb{P}^1 \times \mathbb{P}^1 \longrightarrow \mathbb{P}^1$ is the ith projection for $i = 1, 2$. In case we are working over $k\{\{t\}\}$, one can associate tropical curves Γ_1 and Γ_2 with C_1 and C_2, respectively, just like for curves in $\mathbb{P}^2$.

(1) Convince yourself that the Newton polygons of Γ_1 and Γ_2 are $\mathrm{Conv}\{(0, 0), (d_1, 0), (0, d_2), (d_1, d_2)\}$ and $\mathrm{Conv}\{(0, 0), (e_1, 0), (0, e_2), (e_1, e_2)\}$, and that the Newton polygon of $\Gamma_1 \cup \Gamma_2$ is $\mathrm{Conv}\{(0, 0), (d_1 + e_1, 0), (0, d_2 + e_2), (d_1 + e_1, d_2 + e_2)\}$
(2) Show that the number of intersection points of C_1 and C_2 counted with multiplicities equals $d_1e_2 + d_2e_1$ using tropical geometry.
(3) Show that the vector bundle V is relatively orientable if and only if there are no odd points on the boundary of the Newton polygon of $\Gamma_1 \cup \Gamma_2$.
(4) Show that $\sum_{p\in\Gamma_1\cap\Gamma_2} \mathrm{mult}_p^{\mathbb{A}^1}(\Gamma_1, \Gamma_2) = \frac{d_1e_2+d_2e_1}{2} h \in \mathrm{GW}(k)$ in the relatively orientable case.

Exercise 4.4 (Exercise 1.9 (8) in Maclagen-Sturmfels)**.** Given five general points in $\mathbb{R}^2$, there exists a unique tropical curve with Newton polygon Δ_2 passing through these points. Draw the quadratic curve through the points $(0, 5)$, $(1, 0)$, $(4, 2)$, $(7, 3)$, $(9, 4)$.

Exercise 4.5. Let k be a perfect field of characteristic not equal to 2 or 3 and let $\sigma = (L_1, \ldots, L_s)$ be a list of finite separable field extension of k with $\sum [L_i : k] = 3d - 1$. Recall that

$$N_{d,\sigma}^{\mathbb{A}^1} = \sum \mathrm{Tr}_{\kappa(C)/k}(\mathrm{Wel}_{\kappa(C)}^{\mathbb{A}^1}(C))$$

where the sum goes over all plane rational curves C through a point configuration of s points in general position with residue field $L_1, \ldots, L_s$. Show that if $k = \mathbb{R}$ and $\sigma = (\mathbb{R}, \ldots, \mathbb{R}, \mathbb{C}, \ldots, \mathbb{C})$ with n_2 times $\mathbb{C}$, it holds that

$$\mathrm{sgn}(N_{d,\sigma}^{\mathbb{A}^1}) = W_{d,n_2}$$

that is Welschinger's signed count of real curves.